

DOCUMENT RESUME

ED 243 694

SE 044 465

AUTHOR Hynes, Mary Ellen, Ed.
TITLE Topics Related to Diagnosis in Mathematics for Classroom Teachers.
INSTITUTION Research Council for Diagnostic and Prescriptive Mathematics, Kent, OH.
PUB DATE [78]
NOTE 93p.; Papers presented to the RCDPM Annual Conferences (4th, College Park, MD, April 1-2, 1977) and (5th, Scottsdale, AZ, April 1978). For related documents, see SE 044 464, SE 044 473, and SE 044 479.
AVAILABLE FROM Research Council for Diagnostic and Prescriptive Mathematics, 441 Beryl Drive, Kent, OH 44240.
PUB TYPE Speeches/Conference Papers (150)
EDRS PRICE MF01/PC04 Plus Postage.
DESCRIPTORS Addition; *Diagnostic Teaching; Elementary Education; *Elementary School Mathematics; *Grouping (Instructional Purposes); Learning Problems; Manipulative Materials; *Mathematics Education; *Number Concepts; *Remedial Mathematics; Subtraction

ABSTRACT

This set of papers was selected from presentations made at the Fourth and Fifth Annual Conferences of the Research Council for Diagnostic and Prescriptive Mathematics because of their pertinence to classroom teachers. The papers concern: (1) "Research and Development Related to Learning about Numerals for Whole Numbers: Implications for Classroom/Resource Room/Clinic" (Ashlock); (2) "Levels of Maturity in Diagnosis and Instruction" (Engelhardt); (3) "Teaching Arithmetic: A Diagnostic Approach" (Hopkins); (4) "Using Manipulative Aids to Model Algorithms in Remedial Situations" (Hynes); (5) "A Theoretical Study of the Organization of Basic Addition Facts for Memorization" (Heddens); (6) "How Do You Get Children to Quit Using Their Fingers to Add and Subtract?" (Rathmell); (7) "But I Have 42 Students in My Class" (Connelly); (8) "Implementing Diagnosis in the Classroom" (Inskeep); and (9) "The Role of Dual Programming in Diagnostic and Prescriptive Mathematics" (Wyer). (MNS)

* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *

ED243694

**RESEARCH COUNCIL
FOR
DIAGNOSTIC AND PRESCRIPTIVE MATHEMATICS**

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)

✓ This document has been reproduced as
received from the person or organization
originating it.

☐ Minor changes have been made to improve
reproduction quality.

• Points of view or opinions stated in this docu-
ment do not necessarily represent official NIE
position or policy.

**TOPICS RELATED TO DIAGNOSIS
IN MATHEMATICS
FOR CLASSROOM TEACHERS**

Edited
by
Mary Ellen Hynes
Seminole County, Florida

"PERMISSION TO REPRODUCE THIS
MATERIAL HAS BEEN GRANTED BY

Carol Larson

TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)."

SE044465

TABLE OF CONTENTS

Research and Development Related to Learning About Numerals for Whole Numbers: Implications for Classroom/Resource Room/Clinic	1
<i>Robert B. Ashlock, University of Maryland</i>	
Levels of Maturity in Diagnosis and Instruction	23
<i>Jon Engelhardt, Arizona State University</i>	
Teaching Arithmetic: A Diagnostic Approach	33
<i>Martha Hopkins, Sarasota County, Florida</i>	
Using Manipulative Aids To Model Algorithms in Remedial Situations	41
<i>Michael C. Hynes, University of Central Florida</i>	
A Theoretical Study of the Organization of Basic Addition Facts for Memorization	55
<i>James W. Heddens, Kent State University</i>	
How Do You Get Children To Quit Using Their Fingers To Add and Subtract?	67
<i>Edward C. Rathmell, University of Northern Iowa</i>	
But I Have 42 Students In My Class	75
<i>Ralph D. Connelly, Brock University</i>	
Implementing Diagnosis In The Classroom	79
<i>James E. Inskeep, San Diego State University</i>	
The Role of Dual Programming In Diagnostic and Prescriptive Mathematics	85
<i>George A. Wyer, West Hartford, Connecticut</i>	

Introduction

The following set of papers were selected from the Fourth Annual Diagnostic and Prescriptive Mathematics Conference, held in 1977 at the University of Maryland, and the Fifth Annual Conference held in Scottsdale, Arizona in April 1978.

The purpose of this monograph is to share the ideas of these authors with classroom teachers and other interested readers. The theoretical papers of Engelhardt and Heddens provide the reader with important conceptual frameworks for the classroom-tested ideas of many of the other papers. The Connelly, Inskip, and Wyer papers will be extremely interesting to those teachers who are trying to institute diagnostic and prescriptive instruction in the regular mathematics classroom and find classroom management to be their greatest concern.

**Research and Development Related to
Learning About Numerals for Whole Numbers:
Implications for Classroom/Resource Room/Clinic**

Robert B. Ashlock
University of Maryland

I. INTRODUCTION

I have had an increasing interest in how children learn about and use numerals for whole numbers for I find when working with children in our clinic and in classrooms that it is frequently the case that a child can do specific numeral-related tasks but does not use the numerals as if he or she understood what the symbols represent. This is noteworthy not only when the child computes but also when he or she uses numbers in other contexts. In my work with teachers, I find the term "place value" is used to designate any task which has something to do with what numerals mean, even though place value concepts are only a subset of the set of numeration related concepts. I also find instructional activities are similarly limited...relatively few exemplars are used.

There are also research studies that suggest we need to take a closer look at how children learn and use numerals for whole numbers. Uprichard and Collura (1977) report a clinical study in which an emphasis on mathematical structures such as place value appeared to facilitate not only concepts, but computational skills as well. Charles Smith (1969) noted, when reporting his study involving subtraction computation, that "Pupils proficient on the renaming subtests had less difficulty in subtracting than those less proficient." He found that low achievers on the Iowa Tests of Basic Skills were generally unable to apply decimal numeration principles, and concluded that a greater emphasis needs to be placed on building pupil ability to apply principles of decimal numeration within subtraction computation. However, it should also be pointed out that other researchers, such as DeFlandre (1975) and Fishell (1975), both of whom focused on instruction in numeration and its relation to skill in computation, did not find such understanding necessarily leading to measureable change in computational skill.

Another reason we do well to consider numeration as a topic worthy of our attention is the wide-spread use of hand-held calculators, and the increased need for estimating skill which has resulted. A major prerequisite to estimating skill is understanding of numerals for whole numbers.

II. GENERAL COMMENTS ON RESEARCH

As an area for investigation, children's learning about numerals for whole numbers and their use of numeration principles when doing specific tasks (such as computing or estimating) has not been researched carefully or thoroughly. Much of the research that has been done has focused on learning non-decimal numeration systems, and on the effects of such instruction. But we need to

take a much broader look; we need to consider research across a wider range of issues. In order to do this, we need to more adequately structure the field for investigation; we need to be able to demonstrate the relatedness of the many aspects of teaching and learning numeration. In short, we need theoretical structures which will point more clearly to specific investigations and permit us to relate one study with another more easily.

A. PROBLEMS RELATED TO STRUCTURING THE FIELD OF INVESTIGATION AND FOCUSING RESEARCH

It is not surprising that research in the area has tended to be scattered. Many studies have lacked a clear focus, and even well-done studies have been difficult to relate to one another in terms of implications for instruction.

One difficulty is a lack of consensus as to the definitions of terms, specific areas to be investigated, issues involved, and how these relate to one another. There is much confusion with regard to terminology, not so much among math educators (though the problem exists here also), but especially in the professional literature of other fields of inquiry which bear directly upon our topic. For example, you will find expressions like "recite numerals," or references to oral names which do not make clear what sort of oral name is being used. If we are more careful with our choice of terms we are apt to find this a much more complex area for research and for diagnosis and instruction than many of us have assumed. Perhaps we should not be surprised that our friends from other fields, such as psychology, need help in analyzing the area and in using terminology so it is understood and can be applied by math educators.

I am using the term "number" to refer to the idea or concept of number, the term "whole number" to refer one of a specific set of numbers, and the term "number system" to a system including a set of numbers, operations defined on the set of numbers, a specific properties of those operations. However, I am using the term "numeral" to refer to a standard written symbol made up of figures or digits, and the term "numeration system" to a system for producing numerals. Such systems include a set of figures or digits, and rules for applying concepts such as grouping, place value, addition, and multiplication. There are advantages that accrue when we distinguish between mathematical ideas like number and the symbols we use to write about them. Most notably, they are separate things to be understood. Admittedly many of the errors children make in mathematics result from a lack of understanding of the math itself, as noted by Hamrick and McKillip (1976). But they also concluded from their study that other errors resulted from a lack of understanding of the symbols used.

Perhaps one of the reasons numeration has not been looked at carefully by math educators is that we stand in a very different position than the young child of 6 or 7 who is learning about numerals for whole numbers. As adult

Learning Numerals for Whole Numbers

teachers of mathematics we process mathematical information with comparative ease. Yet Krutetskii (See Kilpartick, 1977) points out that one of the abilities needed to do this is the ability to think in mathematical symbols: both in numerical symbols and letter symbols. We are accustomed to using numerals in thinking about mathematics. But the young child does not yet have the ability to think in mathematical symbols, to use them while focusing on something else. The symbol itself is, at least to some extent, the object of his study. Ginsburg (1977) emphasizes that children search for meaning, for underlying structures even within our systems for naming numbers, a search that sometimes leads to incorrect assumptions for the systems are not thoroughly consistent. My colleague John Elliott, a developmental psychologist concerned with cognition, refers to a symbol system such as a numeration system as a logical system, for it is a system of organization and order with specific conventions; he is concerned that we may be expecting children to learn such logical systems which they have not developed to the point they possess the required logical abilities. Surely, then, we have ample warrant for being concerned with children's understanding of numerals as well as their understanding of numbers and operations on numbers.

But numerals are not the only way we represent whole numbers. Systems of oral number names and the system of written number words parallel the system of numerals. Though we may introduce these systems of representation to children at very nearly the same time, they are different systems. For example, we record the numeral 56 and we mean 5 tens and 6 ones, but when we say the standard number word we do not say 5 tens 6 ones; rather, we say fifty-six. We do not even say 5 tens 6; we say fifty six. Philologists tell us that *eleven* derives from words meaning one left over, *twelve* from words meaning "two over ten," and *hundred* from a term meaning "ten times" (Eves, 1964, p. 8), but such root meanings are, of course, not obvious to children. The systems of representation are obviously related, but they are different. With numbers like 1 ten 3, which we call thirteen, they are very different indeed. Yet, during assessment and instruction, we often use these systems interchangeably, assuming the child can translate easily from one system to the other.

One way of looking at number concepts and the systems of representation is suggested in Figure 1.

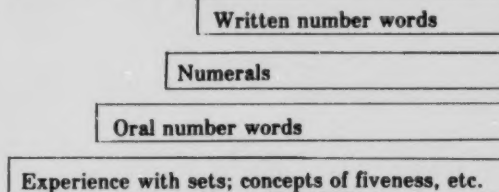


Figure 1 Systems for Representing Whole Numbers

Note that experience with sets occurs first, with number concepts developing from those experiences. Such experiences continue and interrelate with later experiences with representation systems. Of course, children normally do encounter oral number names to twenty or even to a hundred before the corresponding numerals are introduced, for the young child is taught to count by rote. Ginsburg (1977) reminds us that when a child *does* begin to write numerals, he sometimes makes the error of writing numerals in the same way he says corresponding number words. Twenty three might be written 203 as a straight forward translation from words, in accordance with a system of rules which is partly of his own invention.

Such a diagram is an oversimplification, especially for numbers greater than 9, for multi-digit numerals involve concepts deriving from more complex experiences with sets, such as forming sets of sets. They also involve concepts which do not derive from underlying experiences of with sets, such as the concept of value assigned to place.

However, we should be aware that it is possible for a child to make rather straightforward associations of sets with oral words and with numerals for not only 0-9 as suggested in the diagram, but for larger numbers as well. Children who *do* make such straightforward associations can enumerate a set, can tell you how many it contains, and can sometimes write the multi-digit numeral for that number, but they do not understand the structure of the numeral in a way which will permit them to use it creatively when renaming and when estimating. Such children learn about multi-digit numerals much as children learn to read so-called sight words. They work with multi-digit numerals as if they were single digits. In my own experience with children, I have found this happening most often with the teens, with powers of ten, and with multiples of powers of ten. This may be happening when a child writes 10 as the numeral for a set of ten before instruction in grouping and place value. It may also be happening when a child writes 605 for 65, thinking of 60 as a symbol for 6 tens. When children add but fail to rename, they often write answers which crowd two digits into one place; sometimes interpreting their answer correctly without recognizing it makes no sense as a numeral. For example in:

$$\begin{array}{r} 647 \\ +289 \\ \hline 81216 \end{array}$$

the 81216 is interpreted as 8 hundreds, 12 tens, and 16 ones.

Each way of representing numbers is a system with its own structure. For example, within the sequence of oral number words there are patterns which children frequently observe.

Alan was counting by rote. I stopped him after

49 and asked, "What comes next?" "Fifty," he said.

"How do you know?" Immediately he replied, "Cause

Learning Numerals for Whole Numbers

you go 1, 2, 3, 4, 5."

Most children probably observe these patterns on their own, but I have talked with children who are not aware of them. A number of questions occur to me:

What do the children we work with understand about each of these systems of representation as a system?

Do we help children find patterns within each of the systems, or do we expect children to rely only on arbitrary associations or the rules they happen to construct?

Which of a system's concepts do children need to understand? And when?

When are they able to learn them?

We use words as we talk about numerals. When does this confuse the child?

If we "read the numerals" differently than "standard" number words (e.g., "26" as 2 tens 6 ones instead of twenty-six), does this confuse the child when it comes to understanding the systems of word names?

Another difficulty we have when thinking about teaching and learning numeration is deciding what we mean by "understanding," and by "meaning" in this context. Barr (1978) speaks of instruction in the meaning of 2-digit symbols. Fishell (1975) speaks of understanding a decimal numeration system. There are tests called tests of understandings. And so it goes.

What do we mean when we talk of a child knowing or understanding something like place value? From existing research we could abstract a number of operational definitions, and these would vary from specified scores on paper-and-pencil tests to tasks with manipulatives and verbal explanations. Each such behavior lets us infer, with a greater or lesser degree of confidence, that the child understands; he knows. But I have seen children who learned to do many of these specific tasks but were not able to relate the idea to other ideas or to use the information at all; I have seen this happen too often to have much confidence in many of the tasks we give children for evidence of understanding place value or numeration. It is too much to hope the cognitivists and behaviorists will get together on what it means to understand something; but we hope that, in time, they will together give us new insights. Meanwhile, we need to proceed with caution and at least build into research studies a variety of tasks. John Wilson's (1976) grid of possible within-across exemplar class translations from stimulus to response is helpful for planning a variety of tasks, but even these do not explore the full range of sensory modalities that may need to be looked at if we are to find out what a child understands.

B. CATEGORIES OF NUMERALS FOR INVESTIGATION

One thing we can do to help structure the field of study, though it may

seem like a very simple thing, is to agree on categories of numerals for investigation. I suggest three rather self-evident categories; but in research and in teaching, even these distinctions are not always carefully drawn:

1. One-digit numerals (0-9)
2. Two-digit numerals (10-99)
3. Three-or-more-digit numerals ($100 +$)

One-digit numerals are associated with number concepts of "order," "cardinality," and "many-can-be-shown-with-one-digit." The association of number and number name is completely arbitrary; no system of numeration is obvious.

Two-digit numerals do follow a systematic pattern, one which involves new concepts such as "number of groups," "value assigned to a place," "addition," and "multiplication." However, place-value concepts are *not* generalized. Children often think of tens as left and ones as right. It has been encouraging to see a few research studies focusing on 2-digit numerals; those by Rathmell (1973) and Barr (1978) are notable examples. Scrivens (1978) concluded that the third graders he studied had an incomplete understanding of place value beyond 2-digit numerals, and needed more work with tens and ones before extensive work with larger numbers. Smith (1973) concluded that children are too often introduced to the concept of hundreds before adequately understanding the tens in two-digit numerals. These studies recognize 2-digit numerals as a specific category of numerals for research and for instruction: more such studies are to be encouraged.

In contrast to learning about two-digit numerals, when children learn about numerals with three-or-more digits, place-value ideas need to be generalized so the system can be extended indefinitely. John Wilson (1976) has recognized the difference between 2-digit numerals and 3-or-more-digit numerals in his structures of content objectives, specifying additional ideas a child needs to understand when working with numerals of 100 or greater.

I remember Alan, who told me that in the numeral 106, the one is in the tens place and the 6 is in the ones place (he thought of tens as left and ones as right). I asked him, "Where is the zero?" and he said "The zero is in the zero's place." Asked to point to the ones in 2-digit numerals like 26 and 74, Dana said she could not, for "There aren't any ones." Pointing to the digit on the left she said it was hundreds; and then pointing to the other digit she said it was tens. "There aren't any ones." Dana had recently been introduced to three-digit numerals, viewed the digit on the left as hundreds and the next one as tens. She had not yet come to understand the general pattern for assigning place values.

III. DIAGNOSIS

"How do we go about diagnosing what a child understands about numeration?" To begin, we need to consider the full range of abilities and concepts involved in understanding numerals for whole numbers. I want to

Learning Numerals for Whole Numbers

discuss these for a few minutes, then share with you some assessment tasks I have found to be helpful.

A. ABILITIES AND CONCEPTS INVOLVED

A number of attempts have been made to list the concepts and abilities involved in learning about numerals, though such lists sometimes include only content ideas. Examples of lists of concepts and abilities can be found in recently published books by Bitter, Englehardt, and Wiebe (1977), Math H.E.L.P. and One Step at a Time, in Ronshausen's (1978) recent article in The Arithmetic Teacher entitled "Introducing Place Value," and in certain of the content structures constructed by John Wilson (1976). I will not now attempt to present *the* definitive listing of concepts and abilities that might be considered during diagnosis, but I do want to comment about several of them.

Of course, rote counting is involved as a rational counting, and these are skills which entering kindergarteners typically possess to some extent. Much research has focused on a child's understanding of the number of a set; these have been mostly Piagetian studies. But at present the relation of conservation of number to the introduction of numerals is not all clear. Generally, it has been believed that conservation of number is necessary for the development of numerical understanding, but Mpiangu and Gentil (1975) concluded from their study that "Conservation is not necessary for mathematical reasoning." They felt it "would be more profitable to consider arithmetic and conservation of number as conceptualizations that develop simultaneously." I realize this is a very controversial area, and I mention it here only to point out that the place of conservation of number, *per se*, as a concept or ability needed for children to learn about numerals is, as yet, unclear.

Though the earliest written symbols for number were probably tally marks (Wilder, 1968), ours is a cipherized system of numeration; therefore, it is necessary for children to understand that *many* can be shown with *one* symbol, a digit. The face value of each of the ten digits tells how many things. Both research and general practice in our culture support introduction of oral number names before introducing the digits. For the numbers 0-9, the translation is straightforward; difficulties usually begin with the set of ten and above, with the introduction of 2-digit numerals.

For these numerals, it is necessary for children to understand not only the concept of a set, but also the concept of a *set* of sets. Within our cipherized numeration system, this means children need to know not only that many of ones can be shown with one digit, but also that many of tens can be shown with one digit, many of hundreds can be shown with one digit, and so forth.

Use of a base collection is difficult for some children; they can count single objects and can group objects, but they cannot count *groups* of objects. Mary could group objects into sets of five and she could count the number in each set, but she was confused when asked how many sets she had made. Not surprisingly, when shown five tens and two ones with base ten blocks, and

asked how many tens, she said "Seven," though she had previously identified each piece of wood correctly. (Children who do not understand the grouping idea, who *can* count sets of ten, sometimes do not know what to do with the singles left over.) Both Scrivens (1968) and Underwood (1977) concluded from their research activities that the grouping idea is more of a problem for children learning numeration than is the place value idea.

Of course, our numeration system is a place value system, and that is why it lends itself to the processes of arithmetic. Neugebauer (1957, p. 5) considers the invention of place value notation as "undoubtedly one of the most fertile inventions of humanity," an invention that can "be properly compared with the invention of the alphabet as contrasted to the use of picture-signs intended to convey a direct representation of the concept in question."

What do children need to know about place value? They need to know that values, powers of ten, are assigned to each position. Also, they should understand that the position of a digit determines its value; or, as Karen Fuson has said, "What the value of a digit is, depends on where it lives." A related idea, sometimes overlooked, is the fact that only one digit can be assigned to a position or place. When children do not keep this in mind, a variety of errors occur.

Zero is another critical concept. Historically, the digit "zero" was used in several numeration systems as a device to show a place not used, but sometimes this practice was incomplete. For example, the Babylonian zero was apparently used only medially and not at the right of a numeral; 605 and 65 could be distinguished, but not 650 and 65 (from Wilder, 1968). For such distinctions one had to rely on the context, and this has resulted in some interesting dilemmas for archeologists. For us, zero is not only a place holding device, it also names a number. In fact, all ten digits serve both functions; they are place holding devices and they name a number. However, many children do not understand that this is true. For them a zero is merely "nothing." There is a difference between understanding the zero in 204 as "There is nothing in the tens," and understanding it as "There are no tens."

Many other concepts and abilities are involved in learning about numerals for whole numbers. For example, some understanding of both addition and multiplication are required even for two-digit numerals. As John Wilson notes, the value of a digit is the *product* of its face and place values, and the value of a multi-digit numeral is the *sum* of the product values for each digit (Wilson, 1976). There are other abilities and concepts involved in learning about numerals, ones not considered as often by math educators when diagnosing children.

One area of such abilities is spatial orientation and imaging. We have left such concerns to psychologists and neurologists, but when we plan instruction involving written symbols we would do well to consider carefully what they are learning. This is especially true for instruction in multi-digit numerals. We

Learning Numerals for Whole Numbers

are reminded of the difference between one-digit numerals and numerals with two-or-more digits by studies such as those by Luriya (1969), who found that many individuals who become brain damaged retain the ability to perform simple serial computations within the limits of the first ten numbers but are not able to perform more complex operations. He concluded that the basic characteristic of acalculia is impairment of the number concept related to breakdown of thought, which leads to deterioration of number beyond simple numerical relatedness and includes the accompanying loss of place value. We also know from studies such as the one by Guay and McDaniel (1977), that high achievers in mathematics have greater spatial ability than low achievers.

Left-right discrimination is a part of this area, but only part. Children who can identify objects to the left and to the right in relation to a specific object in a sequence, sometimes have great difficulty learning the pattern of assigned place values and the relation of one value to another, either to the left or to the right.

Reversals are often a problem as children begin to write numerals, both the mirror image of a digit and the exchange of two adjacent digits. This phenomenon has been examined more extensively by researchers concerned with language education than by those concerned with mathematics education. Moyer and Newcomer (1977) note that with reference to making the mirror image of a single letter (and we might assume some relevance for making the mirror image of a single digit), two schools of thought exist. One school suggests the difficulty is caused by immaturity of development, the result of neurological malfunctioning, confused lateral dominance; but apparently this has not been verified empirically. The other school suggests there is evidence maturity is not a factor; directionality is a learned cognitive skill. Moyer and Newcomer (1977) suggest that young children use symbols to identify and name things, and it is possible that for them spatial orientation is not a significant feature for identifying and naming things. The child has to learn the importance of this feature.

Interestingly, I recently saw a second grader reversing his 6 so it looked very much like his 2. I dared to show him how a 6 is made and that it is very different than his 2, and he volunteered the fact that he used to make them my way but his friend showed him how to make them the way he was making them. (The next time I observed him making a 6, he made it correctly.)

Stanley (1976) studied children's processing of digits, but they were children with specific reading disabilities. he found that the rounded digits 3, 6, 8, and 9 were most frequently reversed.

Piaget and Inhelder have hypothesized that two-dimensional reference systems do not develop until the period of concrete operational thought, and the findings of Pufal and Shaw (1973) support their hypothesis. It has also been observed that many 7-year-olds can read maps (i.e., use the symbol system to translate symbols into their referents), but they cannot make maps,

they cannot locate and record the symbols in the appropriate place. I am not aware that the implication of this for children being introduced to multi-digit numerals has ever been examined, but it could make a very interesting study.

Another ability which seems to be important in learning about numerals is verbal ability. We know that words are not always associated with correct concepts. Sometimes, the words "more," "greater," and "less" do not mean exactly the same thing to a child as they do to us, but we use these terms regularly during math instruction, even when talking about place values. Order and equivalence relations are certainly involved in learning about numerals, and we tend to assume children have these concepts well in hand long before being introduced to multi-digit numerals. But is this true? Walker (1975) found that initially a child does not use such relational terms in a true comparative sense. Rather, they are used in a nominal sense. The child who uses the term "more" in a nominal sense has difficulty understanding it is possible for the digit in the tens place to be worth *more* than the digit in the ones place, and at the same time be worth *less* than the digit in the hundreds place. Also consider the situation when more milk is added to Joe's glass, but less is added to Mary's glass than to Joe's. McNeill notes in The Acquisition of Language that some children think "less" means "more".

Word order creates some interesting problems, too. I found that what June, a second grader, understood as the meaning of "2 more than," if what I would call "more than 2." (Sets which were "2 more than" a set of 3 included sets of 3, 4, 5, etc.) We have already noted that the word-symbol systems for naming numbers are different from the numeral system, thereby complicating instruction and learning substantially. I think we need to ask: To what extent are children taught what multi-digit numerals mean (and to what extent *should* children be taught what multi-digit numerals mean) *before* they are capable of concrete-operational thought?

B. ASSESSMENT TASKS

If our diagnostic activities with a child are going to get at abilities and concepts such as those which have been described, we will have to have assessment tasks which are different than the usual tasks we find on available instruments. Let me share with you just a few tasks I have found useful in diagnosis.

1. Show: 27 Objects (alike)

Say: "If you want to tell your friend how many are here, is there anything you could do so you would not have to count them one-by-one?
....not have to count them 1, 2, 3, . . . ?

Some children do not group, others form groups but not of the same number, and others form groups of the same number. For those who do not group, I ask "Would this help?" and start grouping by 2's. If more guidance is needed, I

Learning Numerals for Whole Numbers

say "Have you ever counted, 2, 4, 6....? If a child does form groups of the same number, I ask "Could you do it a different way?" Note what a child does with left-over singles. Some children count the singles as if they were groups.

2. Show: 27 objects (alike)

Say: "Make a set of 5 _____"

(objects)

Say: "Now make another set which is 3 *greater* than this set." (point to set child made)

If a child cannot do this, say "Make another set which is *one greater* than this set. If he does, ask for "3 greater" again.

Remove the sets, except the set of 5.

Say: "Now make a set which is 3 *times greater* than this set."

If the child cannot do this say, "Show a set which is 2 times greater than this set."

Children who can easily make a set of 5 often have difficulty making a set which is 3 greater than the set of 5; they make a set of 3. The same results follow with "3 *more* than." The number name is related to the size of the set, and not to the size of the difference of two sets.

For "3 greater than," other children make a set of 4, because "4 is greater than 3." For them, "3 greater than" and "greater than 3" mean the same.

Our instruction often focuses on "greater than;" should we also be focusing on "how many greater than?"

In order to generalize the numeration pattern, children need to be able to think about "N *times greater*." In the numeral 22, one digit is "ten times greater" in value than the other.

3. Show: 5 bundles of ten sticks, and 15 single sticks.

Say: "What would be a good way to show me 26?"

After the child responds....

Say: "How do you know that is 26?"

"Why is the 2 here (point to the digit 2) and not here (point to the 6)?"

Dana readily picked out two bundles of ten and six sticks. In response to the question, "Why is the 2 here?" she said, "Because the ones come first. My teacher says 'Ones comes first.'" Asked, "Which is the ones" Dana pointed to the 2. "Why is it ones?" "Because it is lower." I pointed to the bundles and sticks and asked, "Which is ones?" She pointed to the single sticks. "How many ones?" "6" I pointed to the numeral again and asked "Which is ones?" "2" she said. For Dana, numerals were not a record of observations with sticks or blocks.

4. Show: 17 and 71

Say: "How are they alike?"

"How are they different?"

Ashlock

"Which shows more?" . . . "Why?"

I find this item provides a very effective context for discussing the attributes of a numeral with a child, and it evokes some interesting comments. Some children admit, quite frankly, they do not know which shows more.

5. Show: Pictures of base ten blocks and a 3-digit numeral, as follow:

- a. No zero in numeral, blocks in sequence.
- b. With zero in numeral, blocks in sequence.
(See Appendix A)
- c. No zero in numeral, blocks out of sequence.
(See Appendix B)
- d. With zero in numeral, blocks out of sequence.

For each say: "Draw rings around enough blocks to show as many as this." (point to the numeral!)

After each response say: "Now read this number for me."

I find, not surprisingly, that children who are not consistently able to name the value assigned each place have difficulty relating a digit to the specific piece of wood as they proceed through the tasks. For such children, the zero may not be a problem at all. This item, and a 4-digit variation, do appear to help some children learn to read the numerals.

6. Show: Problems as follows:

- a. $23 + 41 = \underline{\hspace{2cm}}$
- b. $35 + 27 = \underline{\hspace{2cm}}$

For each say: "See if you can figure out the answer in your your head, without writing out the problem."

After each response, say: "How did you add it?"

My interest with items such as this is not in addition skill, as such, but rather what a child understands and how he can use what he knows about numerals. This is an especially interesting question to pose to a child who has *not* been introduced to addition with regrouping. I have been pleased to see the number of children who do add ones to ones and tens to tens when presented with the horizontal format, even when they are not able to read all of the numerals correctly. But an oral response to $36 + 27$ as 513 is not unusual.

There are a number of sources one can turn to, to develop items for assessing numeration. Brownell's (1941) article in the 16th NCTM yearbook included items like "Rearrange the figures in the number 51, 937 to make the smallest possible number," and "In the number 7, 463, what is the largest possible number of hundreds?" More current are selected items in the Maryland Diagnostic Arithmetic Test and Interview Protocols (Wilson and Sadowski, 1976) and Wiebe's (1977) Place Value Knowledge Survey, which includes a number of tasks using quite a variety of materials.

Learning Numerals for Whole Numbers

IV. INSTRUCTION

Let us turn to instruction in numeration and consider a number of ideas, many of which follow from the observations we have discussed.

First of all we ought to recognize that learning about our numeration systems takes time . . . years (Ginsburg 1977). We should not expect children to apply mature concepts of numeration quickly; and when we do introduce computational procedures, we need to carefully guide children in their application of numeration principles. Apparently, numeration is more complex and more difficult for many children than many have assumed. E.g., a child's ability to conserve the numerosness of a set may be critical to the success of instruction in multi-digit numeration.

Also, we should be careful to teach addition with sums no greater than 9 before introducing 2-digit numerals. John Wilson makes a point of this in the hierarchical ordering of his structures of content objectives. If children do not understand the operation of addition, they cannot understand 24 as $20 + 4$ except as an arbitrary manipulation of symbols you do when the teacher gives you blanks to fill in.

Multiplication and division with one-digit numbers should also precede work with ones and tens. Underwood (1977) has made this point, and I believe he is correct. Symbolism need not be introduced, but children need to understand multiplication as we find it suggested in statements like "2 threes" and in "5 tens;" and they need to understand division as we find it suggested in "How many 2s in 6?" and in "How many tens in 34?" The latter even includes the notion of a remainder.

This leads very naturally to an emphasis on the grouping idea, including (eventually) groups of groups. Teachers have not led children to examine such structures until rather recently, and even now it is not clear that groups of groups receive much attention. Certain stick-bundling activities, work with sets of blocks representing powers of ten (and other bases too), and chip-trading activities involve such structures. What is not yet clear is how such activities should be sequentially related to the introduction of oral number names and written multi-digit numerals.

There is considerable controversy at present concerning the introduction of 2-digit numerals from 10 or 20 to 99. The controversy concerns the sequencing of two types of instructional activity: 1) helping the child understand the two digits in terms of the number of groups of ten and the number of units, and 2) helping the child state the standard oral number names in sequence. As far as the numbers 20—99 are concerned, there seems to be a bit of a trend toward teaching the number names for 20—99 through counting activities before teaching the meaning of these particular numerals. If we consider instruction in language as did Hickerson (1952, 1959), we will probably want children to have experiences with sets 20—100 which they count with the standard oral name, to be followed by introducing the symbols

to be read with their standard name, to be followed by writing the symbol. Meaning for the symbols could be developed during the original counting activity (as sets of ten are completed successively and later as numerals are analyzed in terms of tens and ones. An emphasis on counting by tens is likely to help with the often difficult translations back and forth between the standard oral name and the grouping number name for multiples of ten (e.g. "Twenty" and "Two Tens"). Experiences with both rote and rational counting by tens can be provided, and both standard and grouping number names can be used in these experiences.

The familiar counting frame is an excellent exemplar for 2-digit numeration, and it can be used in varied ways. One variation is to count not only with standard oral names but also with grouping number names (e.g., nine, ten, ten and one, ten and two...(later)...5 tens, 5 tens and one, 5 tens and 2, etc.) The other variation is to introduce activity which bridges carefully to the numeral. As one child counts slowly with standard number names, other children can show the numeral with digit cards, or perhaps they can write it. Similarly, as a child counts with grouping number names, other children can show the corresponding numeral.

As 3-digit numerals are introduced, another important question faces us. Is it easier for a child to understand the general principle of numeration from concrete experiences of grouping with a variety of bases, and then apply this understanding to base ten; or to learn the specific pattern of base ten, and then develop a more generalized concept of power? Dienes (1966) argues for the former, but many have concluded from research which has been done on teaching non-decimal number bases that direct teaching of base ten is as effective or better than teaching non-decimal bases. *per se*. Many of the writings of Dienes, and instructional activities such as chip trading games, focus primarily on helping a child abstract the structure of our system. The effects of chip trading games on learning decimal numeration is a proper object of study, but such studies should not be confused with studies which are typical of those focusing on instruction in nondecimal bases.

Instruction in other number bases has been the focus of a number of studies which, taken as a whole, do not give much support for teaching non-decimal bases in order to help children perform more successfully with decimal numeration. Such studies tend toward non-significant findings. When considering such studies it might be helpful to distinguish between numeration systems with the Hindu-Arabic pattern and other systems on the one hand, and decimal and non-decimal bases on the other hand. Four categories are identified in Figure 2:

Such a categorization may help us organize the results of such research in a way that points more clearly to implications for instruction. E.g., in Shrivens' (1968) study, A only was more effective than B followed by A, but he also found instruction in C more effective than B. McCormick (1965) found a

Learning Numerals for Whole Numbers

		BASE	
		Decimal	Non-Decimal
Structure	Hindu-Arabic	A. Our Numeration	B. Our numeration adapted to other bases
	Other Systems	C. Egyptian	D. Mayan

Figure 2: Categories of Numeration Systems

trend toward A only as being more effective than A and B combined.

Another problem area for instruction is the introduction of regrouping. By introducing addition and subtraction of multi-digit numbers without regrouping, and practicing that skill for a year or so (as is typical in our primary programs) children rather typical come to view numerals within examples as a collection of digits to be examined in vertical columns. What they know or may have known about multi-digit numerals tends to be set aside; the idea that in subtraction the minuend is larger than the subtrahend is irrelevant, for they only look at pairs of digits. Perhaps it would make more sense to use what children are learning about numerals as they move into multi-digit addition and subtraction including, from the outset, both examples with *no* regrouping and examples *with* regrouping. Of course, this should be done only by using sticks, base blocks, etc. quite extensively, and delaying introduction of the written algorithm. Answers to given examples would be worked out with materials initially, and the more efficient paper-and-pencil procedures would come later. Though such a suggestion is at variance with current practices in the primary classrooms of American schools, it is not really such a unique idea; and it does potentially allow a child to reinforce what he has learned about numerals and see how that knowledge can be applied, rather than to forget that knowledge and or come to view it as irrelevant with regard to computational procedures.

As we provide children with 3-dimensional exemplars for numeration concepts, there are several cautions we do well to keep in mind. For one thing, we need to be alert to specific limitations of individual children. A child who does well with gross motor activities but less well with fine motor activities should not be asked to bundle sticks; he may be able to do a similar activity with base blocks or an over-sized version of them. Knaupp's (1970) study reminds us that whenever materials like sticks and blocks are used for learning

about numeration, children (except high-IQ children) appear to profit from teacher demonstrations along with their own manipulation of the materials. Of interest also is Wiebe's (1977) finding that during such concrete activities, the amount of pupil verbalization did not have a significant effect on learning the concept of place value.

Steinberg and Anderson (1973) remind us of the presence of distractors in the physical models we use to exemplify numeration concepts. It is true that concrete materials are not helpful just because they are concrete; all such materials are imperfect aids to instruction in that they have attributes irrelevant to the idea being exemplified...color or shape or woodenness or some such. They also vary in the extent to which they exemplify the set of concepts a child needs in order to understand and use numerals.

A categorization of instructional aids may suggest an appropriate order for introducing numeration exemplars. Ashlock (1976) provided the following categorization:

Category 1 "Objects...placed in sets of ten:" examples include stacking chips, bundling sticks, etc. As valuable as such activities may be, we should not be surprised when children do not abstract place value ideas; place value concepts are not exemplified in the activities.

Category 2 "Objects...traded for a larger object which looks like a set of single objects glued together:" examples include "sticks used in a way where there is no bundling but only trading for already-made bundles," base ten blocks which are scored, etc. Knaupp (1970) inferred from his study that the block model for numeration was more interesting and enjoyable for children than the stick model (though in 1970 it may have also been more novel for them), but we should keep in mind that even activities with block models may not exemplify place value concepts. The value of the block is not determined by where it is placed.

Category 3 "Objects which are traded for a single object the same size and shape as a unit object, but are distinguished from the unit object by color or by both color and place:" examples include many chip-trading activities, a computing abacus with different colors, etc. The presence of color makes it impossible to tell if a child has understood and is or is not using the idea that place determines value. And for this reason some have said that such activities should not be used. I think they have their place within a sequence of activities but we should not abandon the child to symbols after Category 3 activities.

Category 4 "Objects traded for a single object identical to the unit object but distinguished from the unit object *only* by the placement of the object:" examples include many place-value charts and devices, a computing abacus with discs of only one color, chip trading games with one color, etc. The child who uses these aids correctly is applying place value ideas and may be

Learning Numerals for Whole Numbers

ready for more extensive work with numerals alone. "Instruction in numeration often involves children with aids in Category 4 too early, and children merely learn complex mechanical procedures for getting answers. The use of varied instructional aids in a sequence similar to that suggested...may do much in helping children relate numerals to appropriate collections in their environment."

V. CONCLUDING COMMENTS

As I close, I would raise again the question: What are we expecting of children in the primary grades with reference to numeration? Arbitrary but unrelated associations? Specific fill-in-the blank tasks? What should we be expecting?

Vygotsky observed that "the young child thinks by remembering, an adolescent remembers by thinking" (Luria, 1976), a statement reflecting thinking which is similar to what we find in Skemp's distinction between instrumental and relational understanding. The young child remembers a specific fact or rule then uses it, whereas the older child uses his understanding of relationships to bring to mind specific facts and rules.

In order for a child to use numerals for whole numbers effectively in computing, in estimating, and in problem solving generally, the child must have a *relational* understanding of such numerals. If we are wise, we will carefully plan our diagnoses, our prescriptions, and our instruction to help children go beyond an instrumental understanding of numerals to a relational understanding.

References

- Barr, David, "A Comparison of Three Methods of Introducing Two-Digit Numeration," Journal for Research in Mathematics Education, 9:33-43, January 1978.
- Beckwith, M., and Restle, F., "Process of Enumeration," Psychological Review, 73, 1966, pp. 437-44.
- Brownell, William A., "Evaluation of Learning in Arithmetic," Chapter X in Arithmetic and General Education, Sixteenth Yearbook of the National Council of Teachers of Mathematics, 1941, 225-267.
- Bruni, James V., and Silverman, Helene, "Developing the Concept of Grouping," The Arithmetic Teacher 21:474-79.
- Chalfant, J. C., and Scheffelin, M.A., "Quantitative Language," Chapter 10 in Central Processing Dysfunctions in Children: A Review of Research. Washington, D. C.: U.S. Department of Health, Education, and Welfare, 1969, pp. 119-131.
- Clark, H.H., "The Primitive Nature of Children's Relational Concepts. In J. R. Hayes (Ed.), Cognition and the Development of Language. New York: Wiley, 1970.

- DeFlandre, Charles, Jr., "The Development of a Unit of Study on Place-Value Numeration Systems, Grades Two, Three, and Four; DAI 35A: 6434; April 1975 (74-19, 747)
- Dienes, Z. P., Building Up Mathematics, London: Hutchinson Educational Hutchinson Educational Ltd., 1960.
- Dienes, P., Mathematics in the Primary School, Rev., London: Macmillan & Co. Ltd., 1966.
- Dienes, Z. P., and Golding, E. W., Approach to Modern Mathematics, New York: Herder and Herder, 1971.
- Dienes, Z. P. and Golding, E. W., Modern Mathematics for Young Children, New York: Herder and Herder, 1966.
- D'Mello, Sydney, and Willemsen, Eleanor, "The Development of the Number Concept: A Scalogram Analysis," Child Development 40:681-688, September 1969
- Eves, Howard, An Introduction to the History of Mathematics, Rev. Ed. New York: Holt, Rinehart and Winston, 1964.
- Fishell, Frank E., "The Effect of a Math Trading Game on Achievement and Attitude in Fifth Grade Division," DAI 36A: 3382; Dec. 1975.
- Flournoy, Frances, "A Study of Pupils' Understanding of Arithmetic in the Primary Grades," Arithmetic Teacher 14 (October 1967): 481-85.
- Gallagher, J. J., The Tutoring of Brain-Injured Mentally Retarded Children, Springfield, Ill.: Charles C. Thomas, 1960.
- Gholam-Hosseini Shokoochi, "Manipulative Devices for Teaching Place Value," The Arithmetic Teacher 25:49-51, March 1978.
- Ginsburg, Herbert, Children's Arithmetic: The Learning Process, New York: D. Van Nostrand Co., 1977, 197 pp.
- Guay, Roland B. and McDaniel, Ernest D., "The Relationship Between Mathematics Achievement and Spatial Abilities Among Elementary School Children," Journal for Research in Mathematics Education 8: 211-215, May 1977.
- Hamrick, Katherine B., and McKillip, William D., "An Investigation of Readiness for the Written Symbolization of Addition and Subtraction at the First Grade Level," Mathematics Education Information Report: Research Sections, NCTM 55th Annual Meeting, Columbus: ETIC-SMEA, December 1976, pp. 43-45
- Hickerson, J. Allen, Guiding Children's Arithmetic Experiences, Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1952, 322 pp.
- Hickerson, J. Allen, "Similarities Between Teaching Language and Arithmetic," The Arithmetic Teacher 6: 241-244, November 1959.
- Kilpatrick, Jeremy, Research on Teaching Mathematics to the Elementary School Pupil, Paper presented at the May 1977 Math Conference, Institute for Research on Teaching, Michigan State University. (unpublished draft)
- Knaupp, Jonathan E., "A Study of Achievement and Attitude of Second Grade

Learning Numerals for Whole Numbers

- Students Using Two Modes of Instruction and Two Manipulative Modes for the Numeration System," DAI 31A:6471 (71-14, 832)
- Lackner, Lois M., "What About Numeration Systems at the Primary Level?" School Science and Mathematics 74:152-156, February 1974.
- Luria, A.R., Cognitive Development: Its Cultural and Social Foundations. Harvard, 1976.
- Luriya, A. R., "On the Pathology of Computational Operations," Soviet Studies in the Psychology of Learning and Teaching Mathematics, ed. by Kilpatrick, J. and Wirsup, I., School Mathematics Study Group and University of Chicago, 1969.
- Manning, Edna McDuffie, "A Study of the Teaching of the Concept of Integers as Directed Distance in Grades Three, Four, and Five and Its Relationship to Piagetian Developmental Level, Left-Right Conceptualization Level, Age, and Achievement," DAI 36A: 5897-5898; March 1976 (76-6054)
- McCormick, Roy Lee, "A Comparative Study of Two Methods of Teaching a Decimal System of Numeration," DAI 26: 1430-1431, Sept. 1965: (65-8637)
- McLeod, Terry M., "The Relationship of Visuo-Spatial Skills and Verbal Ability to Learning Disabilities in Mathematics," DAI 36A: 7991-7992; June 1976. (76-13, 925)
- McNeil, D., The Acquisition of Language, New York: Harper and Row, 1970
- Moyer, Sandra and Newcomer, Phyllis, "Reversals in Reading: Diagnosis and Remediation," Exceptional Children, April 1977, pp. 424-429.
- Mpiangu, Benayame D., and J. Ronald Gentile, "Is Conservation of Number a Necessary Condition for Mathematical Understanding?" Journal for Research in Mathematics Education 6:179-192, May 1975.
- Muzzio, Timothy C., A Study of the Acquisition of the Relational Terms More and Less, Master of Arts Thesis, University of Maryland, 1974.
- Neugebauer, O., The Exact Sciences in Antiquity, 2nd. ed., Providence, R. I., Brown University Press, 1957.
- Payne, Joseph and Rathmell, Edward, "Number and Numeration," Mathematics Learning in Early Childhood, ed. Joseph Payne, Reston, Va.: NCTM, 1975, pp. 125-159.
- Pufall, Peter B. and Shaw, R. E., "Analysis of Development of Children's Spatial Reference Systems", Cognitive Psychology, 1973, 5, 151-175.
- Rathmell, Edward C., "The Effects of Multibase Grouping and Early or Late Introduction of Base Representations on the Mastery Learning of Base and Place Value Numeration in Grade One," DAI 33 A: 6071-6072, May 1973 (73-11, 237)
- Rea, Robert and Robert Reys, "Mathematical Competencies of Entering Kindergarteners," The Arithmetic Teacher 17: 65-74, January 1970.
- Resnick, Laurie B., "Task Analysis in Instructional Design: Some Cases from Mathematics," pp. 149-198 in Schriftenreihe Des IDM, Universitat Bielefeld, Institut fur Didaktik der Mathematik, May 1975. ed. H. Bauersfeld, M. Otte, and H.G. Steiner, 1975.
- Richardson, Alsn, "The Meaning and Measurement of Memory Imagery,"

Ashlock

- British Journal of Psychology, 68:29-43, 1977.
- Ronshausen, Nina L., "Introducing Place Value", The Arithmetic Teacher, 25: 38-41, January 1978.
- Scrivens, Robert W., "A Comparative Study of Different Approaches to Teaching the Hindu-Arabic Numeration System to Third Graders." DAI 29A:839-840, September 1968, (68-13, 400)
- Smith, Charles W. Jr., "A Study of Constant Errors in Subtraction and in the Application of Selected Principles of the Decimal Numeration System Made by Third and Fourth Grade Students," DAI 30: 1084-A, September 1969. (60-14, 685)
- Smith, Robert F., "Diagnosis of Pupil Performance on Place-Value Tasks," The Arithmetic Teacher 20:403-408, May 1973.
- Stanley, G., "The Processing of Digits by Children with Specific Reading Disability (Dyslexia)," The British Journal of Educational Psychology, v. 46, pt. 1, February 1976, pp. 81-84
- Steinberg, Esther R., and Anderson, Bonnie C., "Teaching Tens to Timmy, or A Caution in Teaching with Physical Models," The Arithmetic Teacher 20: 620-25, December 1973.
- Underwood, Dale, "Developmental Concerns Related to Place Value," Presentation at annual meeting of National Council of Teachers of Mathematics, 1977.
- Uprichard, A. Edward, and Collura, Caolyn, "The Effect of Emphasizing Mathematical Structure in the Acquisition of Whole Number Computation Skills (Addition and Subtraction) by Seven and Eight-year olds: A Clinical Investigation," School Science and Mathematics 77:97-104, February 1977.
- Van Engen, Henry, "Place Value and the Number System," in Arithmetic 1947, edited by G. T. Buswell, pp. 59-73. Supplementary Educational Monographs, no. 63. Chicago: University of Chicago Press, 1947.
- Walker, Alice, "The Development of Semantic Components of Number," paper presented at the annual meeting of the American Education Research Association, Washington, D.C., March 30-April 3, 1975. ETIC 1-6 147.
- Wang, Margaret C., Resnick, Lauren B., and Boozer, Robert F., "The Sequence of Development of Some Early Mathematical Behaviors," Child Development 42:1767-1778, December 1971.
- Wiebe, James H., "A Comparison of the Effects of Two Levels of Pupil Verbalization in Learning a Concrete Mathematical Concept," Doctoral Dissertation, Arizona State University, May 1977.
- Wilder, Raymond L., Evolution of Mathematical Concepts: An Elementary Study, New York: John Wiley & Sons, 1968.
- Wilson, John, Diagnosis and Treatment in Arithmetic: Beliefs, Guiding Models, and Procedures, 1976. Published privately, and available through Umporium, University of Maryland, College Park.
- Wilson, John W., and Sadowski, Barbara (eds), The Maryland Diagnostic Arithmetic Test and Interview Protocols, Arithmetic Center, University of Maryland, College Park, Maryland, 1976.

Learning Numerals for Whole Numbers

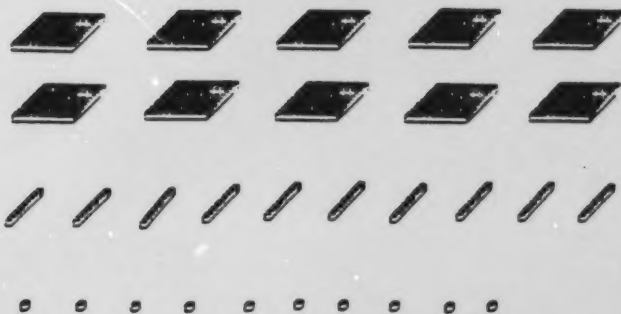
Appendix A

207



Appendix B

436



Jon M. Engelhardt

In helping teachers understand children as learners of mathematics, I often ask them to briefly interview a child on some particular notion of mathematics. During one such interview, a teacher asked a boy named Bill what addition was. Bill's immediate response was to pick up objects and show joining; "You know, like this," he said. When questioned further, Bill was unable (or unwilling) to verbalize any information about addition; he continually responded with joining sets of objects. What is Bill's understanding of addition? Does he *really* understand? Has any information been gained which might aid further instruction on addition?

Given Bill's situation, one group of teachers seem to conclude that Bill knows very little about addition; the fact that Bill did not seem to deal with addition abstractly was convincing evidence of his lack of understanding. These teachers usually suggest that further instruction on addition should proceed from the beginning, assuming very little understanding. Another group of teachers quickly point out that if Bill were a young child his response might not be so bad. Further instruction, for these teachers, would consist of experiences which build upon the child's present level of functioning. Such teachers seem to have an intuitive sense of "understanding" as a dynamic process where dealing with concepts abstractly is the ultimate step.

It is the thesis of this paper that understanding mathematical notions can be expressed through actions, images and symbols and that these alternate expressions of understanding may be conceptualized as levels of maturity. In the paper that follows, the theoretical and experimental bases for this thesis will be explored followed by the implication of the thesis for diagnosis and corrective instruction.

Theoretical and Experimental Bases

Theories of Learning and Intellectual Development

Since the introduction of Pestalozzian teaching methods, about 1800, some educators and psychologists have been talking about how optimum instruction and learning proceed from concrete to abstract. In a summary of principles subscribed to by most modern psychologists, Reys (1971) listed, among others:

1. Learning is based on experience;
2. Sensory learning is the foundation of all experience and thus the heart of learning;

3. Learning proceeds from concrete to abstract; and
4. Learning requires active participation by the learner.

One of the more widely known of modern psychologists is Jean Piaget. Although actually concerned more with epistemology than psychology, Piaget identified four stages of mental growth--stages which inherently stress the concrete-to-abstract nature of learning. According to Piaget, for the first several years of a child's life, mental activity is particularly tied to the active manipulation of objects. At a later age, the child's manipulations begin to be internalized as visual images and ultimately as linguistic, numeric or geometric symbols (Adler, 1966).

The theories of two other modern psychologists have essentially been based upon the views of Jean Piaget. Through his own extensive research, one of these psychologists, Zoltan Dienes (1965, 1967), has evolved four "principles" of mathematics concept learning. Threaded through his explanations of these principles are the notions that, to attain abstract generalization, (1) children should experience concepts in various perceptual settings, including "concrete materials" and "pictures," and (2) concrete settings should precede more abstract settings.

The second psychologist basing his theories upon the work of Jean Piaget is Jerome Bruner. According to the research and theories of Bruner (1960), intellectual development can be characterized in three stages, each distinguishable by the characteristic way in which children view the world and explain it to themselves. Bruner stated that for children events and objects in the world are at first defined in terms of actions taken toward them. Later objects and events are somehow picturable or conceivable without action; images are formed as summarizers of action. Ultimately symbols are used to represent the world, which gradually gain an independence of their own. Describing these stages as systems of representation, Bruner suggested these systems to be mediators through which human thought gains increasing independence from the immediate environment.

Modern psychologists, whose theories are not based upon the ideas of Jean Piaget, have also described learning as stemming from concrete experiences. Galperin (1967), a Soviet psychologist, has postulated learning as the transfer of knowledge from physical action to mental action. Characterizing learning as passing through five states, Galperin suggested that the action is separated from its physical setting and transferred to audible speech, where its basis is transformed from objective to linguistic. Slavina (1967) extended Galperin's ideas to include imagery as a "necessary" intermediate step.

Other psychologists have looked upon learning as the processing and storing of information. In information processing theories, information is stored and transferred between different types of "stores." These stores are distinguished primarily by their temporal properties; however, other properties have also been specified. In a review of multistores theories and

From Physical Activity to Abstract Symbols

research, Craik and Lockhart (1972) reported the three basic stores to be "sensory stores," "short-term stores" and "long-term stores"; most information processing psychologists have agreed that the three stores can be distinguished by the format of information stored--sensory specific (sensory stores), visual/verbal representations (short-term stores) and semantic/linguistic (long-term stores). Although it may be stalled in any store, information is thought to flow from sensory to short-term to long-term stores.

The theories of Piaget, Dienes, Bruner, Galperin and information processing psychologists have been selectively summarized. Although the four principles identified by Reys could be identified, to a greater or lesser degree, within each of the theories, the main emphases were upon the importance of concrete experience and the general concrete-to-abstract progression in intellectual development. A more detailed analysis of these theories, however, leads to three further observations which expand upon the concrete-to-abstract principle.

The concrete-to-abstract nature of general intellectual development pertains to concept attainment as well. Even though the theories of Dienes and the information processors were only sparingly summarized, it is concepts; application of Piaget's and Bruner's theories to specific content, however, is not at first obvious. In applying his learning theory to formulating a theory of instruction, it seems that Bruner (1966) did intend his stages of intellectual development to describe stages of concept attainment as well; for example, he described a teaching procedure for the idea of mathematical group in which the concept was represented concretely through the rotations of a book, graphically through a matrix, and then abstractly through the verbalization of a defining mathematical property.

As for Piaget, referring to the not uncommon example of advanced mathematics students groping for some concrete model of an abstract concept, Reys and Post (1973) suggested Piaget's work implies that in different conceptual areas learners may be functioning at different stages of mental growth. Piaget's theory, like Bruner's, thus appears to have implications (and applications) for concept attainment as well as the more general intellectual development.

Learners can express their understanding of mathematical notions through actions, images and symbols. With the possible exception of information processing theory, the theories of the other psychologists have suggested the importance of object manipulation, visual imagery/pictures and symbolization in representing or promoting mental activity. Similarly, information processors have viewed perception, visual/verbal images and abstract symbolism as differentiating types of memory. It seems reasonable that if learners' mental activity is tied to sensory-motor activity, mental images and symbolisms, then learners should similarly express their level of mental processing through manipulative, graphic and symbolic behaviors.

Engelhardt

Some of the research on human memory, visual imagery research in particular, appears to support this conclusion. For example, in a study of three-year-olds who had been instructed to remember under which of three cups a toy dog was hiding, William, Ritter and Flavell (1975) noted that the children looked toward and touched the to-be-remembered cup more often than the other cups. Such overt behaviors rather obviously indicated the mode of mental activity for these subjects.

In both general intellectual development and concept attainment, sensory-motor activity probably preceeds visual imagery which in turn probably preceeds symbolization. Generally speaking all the summarized theories suggest that the action-image-symbol sequence of mental activity is the natural order in the development of human understanding. This is further supported through research on visual imagery.¹ As Skemp (1971) stated,

But it may well be the case that we all have to go, perhaps more rapidly than the growing child, through similar stages in each new topic which we encounter--that the mode of thinking available is partly a function of the degree to which the concepts have been developed in the primary system. (P. 66)

Bruner the strongest proponent of this developmental order indicated this sequence to be cumulative as well as ordered.

What is abidingly interesting about the nature of intellectual development is that it seems to run the course of these three systems of representation until the human being is able to command all three. (Bruner, 1966, p. 12)

Thus, it appears that either individuals must have experiences representing thoughts with actions, images and symbols or, they must be able to reconstruct such experiences from existent cognitive structures.

In as much as most learning theories appear to have described the evolution of human thought as a gradual disassociation of mental activity from the environment, it seems natural that the ordered action-image-symbol stages of mental activity be conceptualized as "levels of maturity." If this conceptuali-

¹Besides indicating that visual imagery facilitates memory in general (Rowe & Paivio, 1971), research has shown that children above the age of six or seven are able to generate visual images (Levin, Davidson, Wolff & Citron, 1973), while younger children are not (Wolff & Levin, 1972). For these younger children (down to ages three or four) memory is facilitated through relevant motor activity (Wolff & Levin, 1972; Wolff, Levin & Longobardi, 1972; & Levin, McCabe & Bender, 1974). Levin, Ghatala, DeRose, Wilder and Norton (1974) found that by fifth or sixth grade both pictures and words activated visual images. Similar to the theories of general intellectual development, the facilitation of memory appears to proceed from motor activity to the perception of objects/pictures to words.

From Physical Activity to Abstract Symbols

zation is appropriate, then it seems reasonable that a learner can possess concepts without being able to symbolize/verbalize them. Caron (1968) demonstrated this to be true, at least at the visual image level.

The research evidence on the action-image-symbol ordering in concept attainment has not been finally supportive, however. For example, while Scott and Neufeld (1976) found no significant difference between manipulative and pictorial approaches in effecting concept formation, Ekman (1967) reported a manipulative approach to be better than a pictorial one and both to be better than a symbolic approach; more to the point, Punn (1974) indicated that manipulative-pictorial-symbolic and manipulative-symbolic approaches stimulated higher achievement and attitude than a pictorial-symbolic one. As Scott and Neufeld suggested, the considerable variation in terminology makes research conclusions difficult to interpret.

Challenging the entire notion of order, Wittrock (1973) claimed that regardless of whether actions, images or symbols represent or stimulate learning, the important point is the conscious, active involvement (generation) of the individual in learning. He stated that the "learner must actively construct meaning if he is to learn with understanding" (p. 30). To test his claim, Wittrock conducted a brief study with fifth-graders on word definitions; he reported that although generating a picture to help remember each definition was effective, both tracing a picture and copying a definition were ineffective.

In a review of research related to mathematics learning, Bauersfeld (1976) indicated doubts as to whether the various modes of representing concepts actually transmit identical concepts. They may lead to different concepts which appear identical only because of the common symbolisms used to describe them.

Although the action-image-symbol stages for concept attainment and intellectual development appear to be ordered, lack of confirming research and the research and theorizing of Wittrock suggest that this conclusion may be accepted cautiously.

Implications for Diagnosis and Instruction

The thesis of this paper has been that mathematical notions can be expressed through actions, images and symbols, which, in turn, may be conceptualized as levels of maturity. Support for this thesis has been suggested from the psychological theories and base research of Piaget, Dienes, Bruner, Galperin and information processing psychologists. Specifically, it was concluded that besides general intellectual development, specific mathematics understandings (concepts) are attained in a concrete-to-abstract sequence, and the major steps in this sequence--manipulation, visual imagery and symboli-

zation--can be expressed through actions, visual images and symbols. If the steps in attaining understandings are ordered (as they appear to be), then it was suggested that such steps may reflect levels of conceptual maturity.

In the section which follows, the implications of this paper's thesis for diagnosis and instruction will be explored. These implications are not intended to be exhaustive; rather, they represent those procedures which the writer has found useful in a diagnostic/remedial mathematics clinic.

Instruction

The implications of this paper's thesis seem to be more commonly applied to developmental and corrective instruction in mathematics than to diagnostic procedures. Indeed, a number of mathematics educators have attempted to apply the action-image-symbol sequence of concept attainment to mathematics instruction (Underhill, 1972; Reisman, 1972; Heddens, 1974; Sowell, 1974).

The major implication is for sequencing educational experiences. If the development of mathematics concepts naturally proceeds from manipulations to visual images to symbolization, then instruction should be such that it promotes this natural sequence. This is ordinarily achieved through providing a sequence of experiences with manipulative materials, pictures or diagrams, and verbal/numeric symbols. The relative success of the Developing Mathematical Processes mathematics program (Hubbard, 1972) testifies to the appropriateness of this instructional procedure.

Glennon and Wilson (1972) have noted one caution in following this procedure which seems to be particularly appropriate in corrective instruction. They suggested that it is not sufficient to merely experience concrete, pictorial and symbolic materials; they stated:

one learns by *thinking about* what he is doing. And thinking can be done while a person is working with concrete things, or with pictures of things, . . . or with symbols. (P. 309)

In viewing mathematics learning as a generative process, Wittrock (1973) reiterated the importance of not just exposure, but active involvement, with objects, pictures and symbols.

Since general intellectual development appears to proceed from concrete to abstract, the distribution of emphasis upon actions, images and symbols should vary with the learner's developmental level, roughly indicated by age. In the context of *developmental instruction*, the younger the child, the greater the emphasis upon action; the older the child, the less the emphasis upon action and the greater the emphasis upon images and symbols. In the context of *corrective instruction*, this generalization does not appear to be appropriate. Experience in a mathematics clinic has suggested the developmental level within broad content areas to be the crucial factor. Children with a low developmental level in a broad content area like fractions, regardless of age seem to benefit from more prolonged manipulative and image generative experiences.

From Physical Activity to Abstract Symbols

Whether developmental or corrective, instruction at *any age level* appears to benefit from at least some concrete experiences. Archer (1972) for example, found college students to achieve and transfer more when they experienced physical materials in their mathematics instruction.

Experience in a mathematics clinic has further suggested the possibility that the inclusion of special instructional experiences to promote visual images is not always necessary. Many times prolonged experiences with manipulatives appear to be sufficient for some children to succeed in later symbolic experiences. Perhaps this explains why a manipulative-symbolic instructional sequence has sometimes been found to be as effective as a manipulative-symbolic sequence (Punn, 1974). From a pragmatic point of view, instruction might more efficiently follow a manipulative-symbolic sequence, returning to pictorial experiences if the symbolic experiences appear to be too difficult; in fact, experience in the clinic has found this procedure to be very expedient.

Diagnosis

The implications of this paper's thesis for diagnosis in mathematics are somewhat less tenable and not as commonly applied as those for instruction. Crucial to such implications is the action-image-symbol sequence in attaining mathematics concepts. Although there seems to be some support for this sequence; if there is no such order (as Wittrock indicated), then manipulative, pictorial and symbolic expressions of a concept merely suggest the completeness of understanding. For example, if a student can generate equivalent fractions symbolically but not pictorially (or vice versa), then the student would be diagnosed as having only a partial understanding; corrective instruction for such a diagnosis would consist of helping the student display the appropriate manipulative, graphic or symbolic behaviors.

It is a truism that effective instruction must begin where the student is. Assuming the action-image-symbol sequence is ordered, if the student's level of attainment in a given concept is at the manipulative or visual imagery level, then it seems reasonable that instruction should start at that level. Although this is to an extent an implication for instruction, it is dependent upon a diagnosis which identifies the step in the action-image-symbol sequence where the child's concept development ceased. Therefore, the main implication for diagnosis is that a student's performance in manipulative, pictorial and symbolic tasks which embed a given concept should be examined to infer the level of attainment (maturity, if you will).

This procedure would appear to be particularly useful in situations where a child has experienced previous instruction which followed an action-image-symbol sequence. Whether the instruction failed (or ceased) or the child was not ready, the diagnosis suggests (a) where in the sequence instruction or concept attainment ceased and (b) where continued or corrective instruction should begin.

Experience in a mathematics clinic has suggested that examining

Engelhardt

children's performance in manipulative, pictorial and symbolic tasks is in fact, an effective procedure for determining their conceptual maturity. Although there are some exceptions at older ages and in more advanced concept areas, clinic teachers have been able to determine cutoff points in the sequence for different conceptual areas. For example, it is not uncommon for a child to deal with division of whole numbers in manipulative and pictorial tasks, but fail utterly in symbolic tasks.

With respect to implementing the diagnostic procedure outlined above, several questions appear to be in need of attention. First, for a given concept, should diagnoses proceed from concrete to abstract or vice versa? In the former case the diagnosis can become extremely long, while in the latter case initial failure may have an effect upon later performance. This question is related to another unanswered question concerning the relative merits of ascending and descending diagnosis.

A second question in need of further consideration is whether it is possible to find or create manipulative (and possibly pictorial) tasks for diagnosing all mathematics concepts. Clinic teachers who have dealt with older children in the areas of decimals, percents and even fractions, for example, have particularly struggled with this question.

What effect upon performance does student unfamiliarity with the materials employed in the manipulative tasks have? This question seems to be particularly appropriate when dealing with an older child. If a child can perform symbolically but not with manipulatives, is this an indication of unfamiliarity with materials or is this symptomatic of lesser understanding?

Finally, are there ways in which manipulative tasks can be administered to more than one child at a time? Mathematics clinic experience has shown that students seated in a circle at desks or tables facing outward can be easily observed during manipulative tasks, but cannot themselves easily observe other children's behavior. Unfortunately, this procedure has proved feasible only for small groups.

Conclusion

The thesis of this paper has been that mathematical notions can be expressed through actions, visual images and symbols, which suggest levels of conceptual maturity. Support for the thesis was indicated from the theories and base research of several psychologists. Implications for diagnosis and instruction were then explored, and several questions relative to the implied diagnostic procedure were raised.

The ideas presented in this paper are intended to provide a framework for helping those children who experience learning difficulties in mathematics. At the outset, it was the writers intent to challenge the reader with a thesis whose implications related diagnosis and corrective instruction. The real challenge of this paper, however, is to stimulate discussion and research which may ultimately lead to improvement in mathematics education.

Froia Physical Activity to Abstract Symbols

References

- Adler, I. Mental growth and the art of teaching. Arithmetic Teacher, 1966, 13, 576-584.
- Archer, J. A. Effects of concrete, semi-concrete, and abstract teaching methods on mathematical achievement, transfer, and retention at the college level. (Doctoral dissertation, George Peabody, 1972). Dissertation Abstracts International, 1973, 32, 1580.
- Bauersfeld, H. Research related to the mathematical learning process. In H. Athen, & H. Kunle (Eds.), Proceedings of the International Congress on Mathematical Education. Karlsruhe, West Germany: University of Karlsruhe, 1977.
- Bruner, J. S. The process of education. New York: Vintage Books, 1960.
- Bruner, J. S. Toward a theory of instruction. New York: W.W. Norton, 1966.
- Caron, A. J. Conceptual transfer in preverbal children as a consequence of dimensional training. Journal of Experimental Child Psychology, 1968, 6, 522-542.
- Craik, F. I. M., & Lockhart, R. S. Levels of processing: A framework for memory research. Journal of Verbal Learning and Verbal Behaviors, 1972, 11, 671-684.
- Dienes, Z. P. The arithmetic and algebra of natural numbers. Essex, England: Educational Supply Association, 1965.
- Dienes, Z. P. Building up mathematics. London: Hutchinson, 1967.
- Dienes, Z. P. Some basic processes involved in mathematics learning. In J. M. Scandura (Ed.), Research in mathematics education. Washington, D. C.: National Council of Teachers of Math, 1967.
- Ekman, L. G. A comparison of the effectiveness of different approaches to the teaching of addition and subtraction algorithms in the third grade. (Doctoral dissertation, University of Minnesota, 1966). Dissertation Abstracts International, 1967, 27, 2275-2276.
- Galperin, P. I. An experimental study in the formation of mental actions. In B. Simon (Ed.), Psychology in the Soviet Union. Stanford, Calif.: Stanford University Press, 1967.
- Glennon, V. J., & Wilson, J. W. Diagnostic-prescriptive teaching. In W. C. Lowry (Ed.), The slow learner in mathematics. Washington, D. C.: National Council of Teachers of Math, 1972.
- Heddens, J. W. Today's mathematics (3rd ed.). Chicago: Science Research Associates, 1974.
- Hubbard, W. D. Developing mathematics processes: A report of the 1971-1972 field test. (Tech. Rep. 248). Madison: Wisconsin Research and Development Center for Cognitive Learning, December 1972.
- Levin, J. R., Davidson, R. E., Wolff, P., & Citron, M. A comparison of induced imagery and sentence strategies in children's paired-associate

Engelhardt

- learning. Journal of Education Psychology, 1973, 64, 306-309.
- Levin, J. R., Ghatala, E. S., DeRose, T. M., Wilder, L., & Norton, R. W. A further comparison of imagery and vocalization strategies in children's discrimination learning. (Tech. Rep. 299). Madison: Wisconsin Research and Development Center for Cognitive Learning, May 1974.
- Levin, J. R., McCabe, A. E., & Bender, B. G. Imagery induction in the pre-imagery child. (Tech. Rep. 282). Madison: Wisconsin Research and Development Center for Cognitive Learning, July 1974.
- Punn, A. K. The effects of using three modes of representation in teaching multiplication facts on the achievement and attitude of third grade pupils. (Doctoral dissertation, University of Denver, 1973). Dissertation Abstracts International, 1974, 34, 6954-6955.
- Reisman, F. K. A guide to the diagnostic teaching of arithmetic. Columbus, Ohio: Charles E. Merrill, 1972.
- Reys, R. E. Considerations for teachers using manipulative materials. Arithmetic Teacher, 1971, 18, 551-558.
- Reys, R. E., & Post, T. R. The mathematics laboratory: Theory to practice. Boston: Prindle, Weber & Schmidt, 1973.
- Rowe, E. J., & Paivio, A. Imagery and repetition instructions in verbal discrimination and incidental paired-associate learning. Journal of Verbal Learning and Verbal Behavior, 1971, 10, 668-672.
- Scott, L. F., & Neufeld, H. Concrete instruction in elementary school mathematics: Pictorial vs. manipulative. School Science and Mathematics, 1976, 76, 68-72.
- Siegel, A. W., & Allik, J. P. A developmental study of visual and auditory short-term memory. Pittsburgh: Learning Research and Development Center, 1973.
- Skemp, R. R. The psychology of learning mathematics. Baltimore: Penguin Books, 1971.
- Slavina, L. S. Specific features on the intellectual work of unsuccessful pupils. In B. Simon (Ed.), Psychology in the Soviet Union. Stanford, Calif.: Stanford University Press, 1967.
- Sowell, E. Another look at materials in elementary school mathematics. School Science and Mathematics, 1974, 74, 207-211.
- Underhill, R. G. Teaching elementary school mathematics. Columbus, Ohio: Charles E. Merrill, 1972.
- Wellman, H. M., Ritter, K., & Flavell, J. H. Deliberate memory behavior in the delayed reactions of very young children. Developmental Psychology, 1975, 11, 780-787.
- Wolff, P., Levin, J. R., & Longobardi, E. T. Motoric mediation in children's paired associate learning: Effects of visual and tactual contact. Journal of Experimental Child Psychology, 1972, 14, 176-183.
- Wolff, P., & Levin, J. R. The role of overt activity in children's imagery production. Child Development, 1972, 43, 537-547.

Martha Hopkins

The process of developing tests designed to determine specific areas of deficiency in arithmetic computation involves four basic steps:

- (1) Development of the continuum of skills involved in each operation to be tested;
- (2) Identification of the common error patterns made by children while computing;
- (3) Development of criteria for the diagnostic instrument itself;
- (4) Development of a matrix used to analyze the test answers.

Each step will be discussed briefly.

Development of the Continuum of Skills

When learning any arithmetic operation, a child must proceed through a series of small steps beginning with the simplest type of problem encountered in that operation and ending with the most complex. In order to diagnose the placement of a child within the continuum for any particular operation, the teacher (or test developer) must analyze the operation to be tested and determine the most logical sequence of steps through which the children would move when learning that operation. Chart 1 illustrates one possible sequence for learning addition.

ADDITION SEQUENCE

Step 1	Basic facts with sums less than 10.	(7+2)
Step 2	Tens plus one digit without regrouping.	(20+6)
Step 3	Two digit plus one digit without regrouping.	(25+1)
Step 4	Tens plus tens without regrouping.	(30+50)
Step 5	Two digit plus two digit without regrouping.	(43+26)
Step 6	Basic facts with sums greater than 9.	(8+4)
Step 7	Two digit plus one digit with grouping to ten.	(43+7)
Step 8	Two digit plus one digit with regrouping in ones place.	(46+7)

Martha Hopkins

Step 9	Tens plus tens with regrouping in tens place.	(50+70)
Step 10	Two digit plus two digit with regrouping in ones place.	(15+86)
Step 11	Two digit plus two digit with regrouping in tens place.	(75+52)
Step 12	Two digit plus two digit with regrouping in ones and tens places.	(98+76)
Step 13	Addition with more than two addends.	(27+6+14)

CHART 1

Source: Hopkins, Martha H., Teaching Arithmetic: A Diagnostic Approach, 1977.

As one can see, Steps 1—5 represent addition problems requiring only knowledge of the basic facts with sums less than 10. With the mastery of the higher basic facts (sums greater than 9) in Step 6, the child can proceed through steps 7—13 which involve the regrouping process. Many texts include the higher basic facts with the easier ones right away. The sequence in Chart 1 was developed around the theory that a child cannot truly internalize information unless it is used repetitively. Why learn the higher facts before one can actually use that knowledge to solve problems?

Already the reader has seen that there is no one agreed upon sequence of learning the addition operation. Charts 2 and 3 further illustrate the variability of addition sequences.

ALTERNATE ADDITION SEQUENCE--A

Step 1	Basic facts with sums less than 10.	(7+2)
Step 2	Tens plus tens without regrouping.	(30+50)
Step 3	Two digit plus two digit without regrouping.	(43+26)
Step 4	Tens plus one digit without regrouping.	(20+6)
Step 5	Two digit plus one digit without regrouping.	(25+1)
Step 6	Basic facts with sums greater than 9.	(8+4)

Teaching Arithmetic: A Diagnostic Approach

Step 7	Tens plus tens with regrouping in the tens place.	$(50+70)$
Step 8	Two digit plus two digit with regrouping in tens place.	$(75+52)$
Step 9	Two digit plus two digit with regrouping in ones place.	$(15+86)$
Step 10	Two digit plus two digit with regrouping in ones and tens places.	$(98+76)$
Step 11	Two digit plus one digit with regrouping to ten.	$(43+7)$
Step 12	Two digit plus one digit with regrouping in ones place.	$(46+7)$
Step 13	Addition with more than two addends.	$(26+6+14)$

CHART 2

Study of Chart 2 reveals that an alternate sequence would be to introduce ragged addition problems (the addends have unequal number of digits) *AFTER* the child has had experiences (and success) with problems involving addends having the same number of digits. The theory here is that ragged addition problems create error patterns unique to that format ($26 + 2 = 48$) and there is no need to begin the child's experience in addition with a potentially confusing format. (Opponents to this philosophy argue that two digit plus one digit problems are actually at the beginning of the sequence because of the relationship such problems have to the manipulative counting experiences of the early primary years.)

The regrouping problems in Chart 2 reflect this same theory about ragged addition problems being more difficult than those with a consistent number of digits in the addends.

ALTERNATE ADDITION SEQUENCE--B

Step 1	Basic facts with sums less than 10.	$(7+2)$
Step 2	Tens plus tens without regrouping.	$(30+50)$
Step 3	Two digit plus two digit without regrouping.	$(43+26)$
Step 4	Tens plus one digit without regrouping.	$(20+6)$

Hopkins

Step 5	Two digit plus one digit without regrouping.	(25+1)
Step 6	Basic facts with sums greater than 9.	(8+4)
Step 7	Tens plus tens with regrouping in the tens place.	(50+70)
Step 8	Two digit plus two digit with regrouping in tens place.	(75+52)
Step 9	Two digit plus one digit with regrouping to ten.	(43+7)
Step 10	Two digit plus one digit with regrouping in ones place.	(46+7)
Step 11	Two digit plus two digit with regrouping in ones place.	(15+86)
Step 12	Two digit plus two digit with regrouping in ones and tens places.	(98+76)
Step 13	Addition with more than two addends.	(26+6+14)

CHART 3

Chart 3 differs from Chart 2 only in the regrouping steps (7-12). The theory here is that once the children learn the addition process involving addends containing varying numbers of digits (ragged addition) without regrouping, this process will be transferred to those problems requiring regrouping. Based on that assumption, it becomes more logical to proceed with regrouping in the ones, then the tens, and finally in the ones and tens.

Determination of the continuum of skills involved in each operation requires the teacher (or test developer) to decide upon the sequence which seems more logical to him/her or, more importantly, to the child(ren) being tested. Once the sequence has been determined, the test developer is ready to begin with Step 2 of the test development process.

Identification of Common Error Patterns

Determining the approximate placement of the child among the continuum of skills for the operation taught is not a complete diagnosis of the child's arithmetic needs. Now the teacher/test developer must determine *WHY* the child was unsuccessful on the higher steps of the sequence. Are there specific error patterns that show up frequently in the child's work? The examples below illustrate a few of the most common errors children make in addition. The reader is encouraged to try to discover how each child reached his particular answer before reading the explanations following the examples.

Teaching Arithmetic: A Diagnostic Approach

a)	$\begin{array}{r} 23 \\ + 9 \\ \hline 41 \end{array}$	b)	$\begin{array}{r} 23 \\ + 9 \\ \hline 122 \end{array}$	c)	$\begin{array}{r} 23 \\ + 9 \\ \hline 12 \end{array}$	d)	$\begin{array}{r} 23 \\ + 9 \\ \hline 31 \end{array}$	e)	$\begin{array}{r} 23 \\ + 9 \\ \hline 212 \end{array}$
----	---	----	--	----	---	----	---	----	--

Solutions:

- Writes the regrouped figure instead of the remainder. ($3+9=12$, put down the 1 and carry the 2; $2+2=4$)
- Adds ones to ones and tens. ($3+9=12$, put down the 2 and carry the 1; $9+2+1=12$)
- Does not finish the problem. ($3+9=12$)
- Error in basic facts. ($3+9=11$)
- Treats each column as a separate problem. ($3+9=12$; $2+0=2$)

Upon completion of this exercise, the reader no doubt has a better understanding of the importance of testing for common error patterns. The information derived from such tests will allow the teacher to concentrate his/her instructional efforts only on the specific areas of difficulty of the child rather than resorting to reteaching the *ENTIRE* addition process (the latter of which often results in a long term disinterest and frustration with arithmetic in general.) of

Development of Criteria for the Diagnostic Instrument

When developing a test to be used for diagnostic purposes, it is important to decide beforehand such things as: (1) How many items should be included on the test? (2) How should the items be arranged? (3) Are there any pitfalls to avoid when constructing the test items?

Chart 4 illustrates the criteria I used when developing a set of computational diagnostic instruments. Once the criteria has been developed it is of utmost importance that the teacher/test developer not stray from them unless *ABSOLUTELY* necessary. Frequent straying from the criteria set up by the developer will result in an inconsistent and ineffective test instrument; thus, defeating the purpose of the test.

CRITERIA FOR TEST DESIGN

- Each step in the sequence of the operation tested must contain four problems---three in vertical form and one in horizontal. (With only one or two problems per step, the amount of data gathered will be insufficient to insure correct placement along the continuum.)
- The items in each step must be arranged in order of difficulty, with the first problem being the easiest.
- All basic combinations must be used at least once and no more than four times, if possible. This assures that a child's lack of mastery of any partic-

Hopkins

- ular fact will not adversely effect his/her progression through the steps.
4. Problems must be designed so as to negate the possibility of a child making an error and still achieving the correct answer.

Example:

15 The child could regroup the wrong digit (Error #8) and still get the

$$\begin{array}{r} + 6 \\ 21 \end{array}$$
 correct answer.

5. Items must be designed in such a way as to prevent overlapping of incorrect responses as much as possible.

Example:

- 43 a) Did the child regroup from tens to ones without changing the

$$\begin{array}{r} -18 \\ 35 \end{array}$$
 tens digit?
- 35 b) Did the child subtract the smaller digit from the larger digit instead of regrouping?

Source: Hopkins, Martha H., Teaching Arithmetic: A Diagnostic Approach, 1977, p. 11-12.

CHART 4

Development of the Analysis Matrix

Perhaps the most useful part of the diagnostic instruments discussed in this paper is the matrix used by the teacher to analyze the answers computed by the child(ren). Chart 5 illustrates a small portion of the matrix I created for use with an addition test.

Addition Diagnostic Test Analysis Sheet

Problem	Error 1	2	3	4	5	6	7	8	9	10	11	12
9) $\begin{array}{r} 11 \\ +3 \\ \hline 14 \end{array}$	15 etc.	13	4, 1		44		24				8, 33	5
10) $\begin{array}{r} 45 \\ +3 \\ \hline 48 \end{array}$	49 etc.		8, 4		78		58				42, 135	12
11) $\begin{array}{r} 94 \\ +4 \\ \hline 98 \end{array}$	97, etc.		8, 9		138		108				90, 376	17

CHART 5

Teaching Arithmetic: A Diagnostic Approach

Careful examination of the elements of this matrix will help the reader see how extremely convenient and time saving this portion of the test can be. Across the top of the matrix are the numbers 1—12. These numbers represent the 12 error patterns I found to be the most common. (These, of course, need to be explained and illustrated on a separate page of the test materials.) The extreme left hand column of the matrix contains the test items themselves (listed in the same order as found on the testing instrument). The remaining boxes contain the answers children would get if they made each of the common errors identified across the top. For example, if the child's answer to problem #9 had been 24, the teacher could assume that the child *MAY* have difficulty with error pattern 7. I say "*MAY*" because it is extremely dangerous to assume anything from one problem only. If the same child answered 58 for problem #10 and 108 for problem #11, then it would be much safer to assume that instruction should begin with error pattern 7.

One of the advantages of a matrix such as this one is that many teachers have not received adequate training in the area of error pattern recognition; and, therefore, find it quite difficult--if not impossible--to analyze every incorrect answer a child may produce. With the use of matrices such as this one, a teacher needs only to find the incorrect answer and then refer to the explanation of the error pattern that answer falls under.

I would be remiss if I did not also point out one of the dangers of matrices such as this one. Most teachers can tell you that oftentimes a child will get a particular answer (right or wrong) with a thought process quite different from that at which he/she expected. Therefore, it becomes of utmost importance when using an error pattern analysis matrix to *ALWAYS* go back to the child and him/her to do a similar problem while thinking aloud. Then, and only then, be absolutely certain that your assumptions about his/her instructional needs are correct.

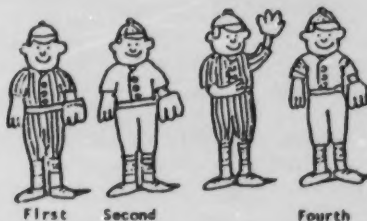
Using these four simple and logical steps makes it possible to design diagnostic tests for virtually any computational area. Tests have been developed in the four major areas of addition, subtraction, multiplication, and division and are presently being completed in the area fractions. Also, I've heard of the same process being applied successfully to areas of secondary mathematics such as Algebra. The involvement of more educators in this diagnostic process should lead to the ultimate goal of daily classroom diagnosis--without the need for specific tests!!

Using Manipulative Aids
To Model Algorithms
In Remedial Situations

Michael C. Hynes
University of Central Florida

Manipulative aids are often used in the primary grades to model basic addition, but the use of these concrete level devices seems to dwindle in the upper grades. The purpose of this paper is to provide the reader with examples of common algorithms illustrated by manipulative aids. Let us begin with the situation of a man entering an elevator and asking the elevator attendant, "Fifth floor, please." The attendant replies, "I am sorry, sir, but the fifth floor button is broken." Quickly the man quipped, "O.K., push one and four, then." This humorous situation typifies a confusion among young children. Cardinality and ordinality of whole numbers seem to blend into one confused mess for some pupils in the primary grades.

Cardinality and ordinality should be illustrated using sets of objects. To illustrate the cardinal nature of whole numbers, sets of objects (popsicle sticks, colored chips, "lots of links", shells, acorns, etc.) can be rationally counted one by one. However, it is helpful when teaching ordinal numbers and the corresponding words for ordinal numbers to use sets of objects that are distinguishable in a set. For example, in the *KEY MATH* test, one of the items shows several ducklings. The first, second, third and fourth ducks are yellow, while the fifth is brown. The children are told the ordinal names of the first three ducks and they are asked the name of the brown duck. This pictorial question implies how the concept of ordinal numbers can be taught using sets of real objects: toys, stuffed animals, colored chips, flowers, etc. The distinguishable objects should be placed in clear view of all the children and the ordinal number names for all but one should be stated to the children.

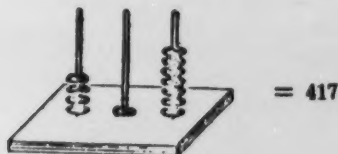


Unfortunately, due to chronic absenteeism and other factors, children often learn isolated and unrelated facts for which they have no conceptual understanding. If place value is learned in a fragmented fashion with little understanding, recurring errors in the algorithms for whole number addition,

Hynes

subtraction, multiplication, and division are possible. There are many manipulative aids which can be used to illustrate the concept of place value, and help children gain a better understanding of this vital topic.

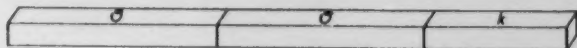
Let's look at the procedure for illustrating place value on an open-end abacus.



The figure above shows how 417 is illustrated on the abacus. The seven beads on the right hand post stand for seven ones, the four beads on the left post represent four hundreds, and the bead on the middle post is a ten. Some children do have difficulty understanding that the beads on the tens post represent ten of the beads on the ones post. The only clue to the fact that they have different values is their position on the open-end abacus. Thus, this manipulative aid may not be the best to use for initial examples of multi-digit numbers.

Children may develop a better understanding of place value if manipulative aids which use stimuli in addition to position to illustrate place value. Size and color are examples of other stimuli which can be utilized and showing place value. For example on the abacus board the ones, tens, and hundreds beads could be different colors. This additional clue to the relationships among the ones, tens, and hundreds could be enough to allow some additional students in a class to grasp the concept being taught. Yet, there may still be children who do not understand place value because they do not understand why one red bead is equivalent to ten white beads.

Centimeter rods can be used to illustrate place value concepts, and an additional stimuli is apparent. A size attribute is added to the color and positional stimuli of the open-end abacus. For example, the number twenty-seven would be shown as:



$$= 27$$

Q = orange; K = black

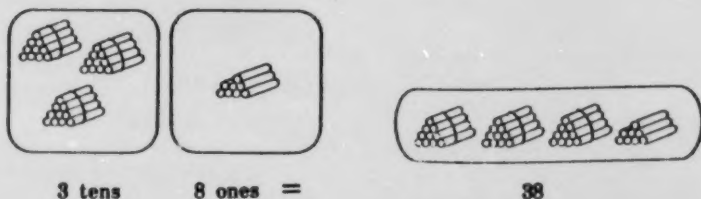
44

Using Manipulative Aids to Model Algorithms in Remedial Situations

These added stimuli help many students gain a better understanding of place value.

The simplest and least expensive aids are often the most effective, and in teaching place value, counting sticks rank very high on the list of useful aids. Counting sticks involve the stimuli of size and position, and if used in conjunction with a place value box, the color stimulus may be added.

By looking at the example below, one can see that the 3 in the tens place represents 3 bundles of tens or 30 popsicle sticks while the 8 represents 8 ones.



Students can visually see that the bundles in the tens place are bigger than the ones, can feel that the tens are heavier, and can count that there are ten sticks in each of the bundles. Thus, the tenness of the 3 in 38 is well established through the many stimuli related to counting sticks.

As was mentioned earlier the stimulus of color may be added to counting sticks by putting the sticks into a place value box. The tens box can be made one color and the ones another.



The place value box not only allows the use of color, but provides some organization for the counting sticks.

Although there is no research evidence as yet to tell what kind or how many stimuli should be presented to students to maximize learning, it is clear, however, that children learn mathematics in many different ways. Therefore, the concrete aids utilized in corrective instruction should include as many visual, auditory or tactile clues as possible.

Hynes

Let us now look at how these manipulative aids can assist the teacher in corrective instruction when error patterns in abstract problems appear on tests, worksheets, or homework. Ashlock (1975) has documented many such error patterns in his book *Error Patterns in Computation*. One of the error patterns he has included from additions is.

48	28	36	73	55
$\begin{array}{r} 48 \\ + 5 \\ \hline 17 \end{array}$	$\begin{array}{r} 28 \\ + 7 \\ \hline 17 \end{array}$	$\begin{array}{r} 36 \\ + 9 \\ \hline 18 \end{array}$	$\begin{array}{r} 73 \\ + 8 \\ \hline 18 \end{array}$	$\begin{array}{r} 55 \\ + 9 \\ \hline 19 \end{array}$

His explanation of this pattern is that the student adds the digits and ignores place value. This diagnosis indicates that more understanding of place value is needed to correct the error pattern. Before we look at how to show these examples concretely, let's look at a similar error pattern documented by Hollingsworth (1977).

48	28	36	73	55
$\begin{array}{r} 48 \\ + 5 \\ \hline 71 \end{array}$	$\begin{array}{r} 28 \\ + 7 \\ \hline 71 \end{array}$	$\begin{array}{r} 36 \\ + 9 \\ \hline 81 \end{array}$	$\begin{array}{r} 73 \\ + 8 \\ \hline 81 \end{array}$	$\begin{array}{r} 55 \\ + 9 \\ \hline 91 \end{array}$

This pattern has an easily recognizable pattern similar to the Ashlock pattern, but there is another pattern. The more apparent pattern indicates that the digits are added 48 ($4 + 8 + 5 = 17$), and the result is reversed, (71).
+5

The less obvious solution is that the ones are added but the ones are regrouped rather than the tens

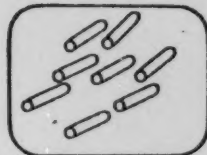
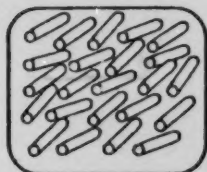
$$\begin{array}{r} 3 \\ 48 \\ + 5 \\ \hline 71 \end{array}$$

Finally the "tens" are added to give the result, 71.

Regardless which error pattern was used by a student, the student had a lack of understanding of how place value is related to the algorithm of addition. An example of a two-digit number added to a single-digit number involving regrouping using some concrete manipulative aids might help overcome this lack of understanding. Using popsicle sticks or other counting objects that can be bundled, the algorithm of addition can be illustrated while emphasizing place value.

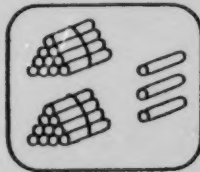
Using Manipulative Aids to Model Algorithms in Remedial Situations

STEP 1



$$\begin{array}{r} 23 \\ + 8 \\ \hline \end{array}$$

STEP 2



$$\begin{array}{r} 2 \text{ tens} \quad 3 \text{ ones} \\ + \quad \quad 8 \text{ ones} \\ \hline \end{array}$$

STEP 3



2 tens

11 ones

=

STEP 4



2 tens

1 ten and 1 one

=

STEP 5



3 tens and 1 one

=

STEP 6

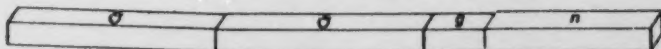


31

Hynes

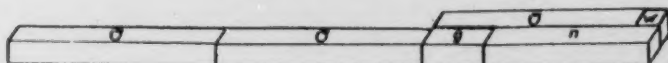
With centimeter rods a similar procedure can be used to illustrate the same example.

STEP 1



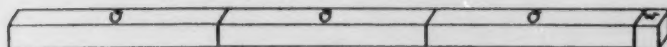
$$\begin{array}{r} 2 \text{ tens } 3 \text{ ones} \\ + \quad \quad 8 \text{ ones} \\ \hline \end{array}$$

STEP 2



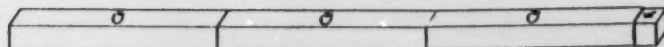
$$\begin{array}{r} 2 \text{ tens } 3 \text{ ones} \\ + \quad \quad 8 \text{ ones} \\ \hline \end{array} \quad \text{and } 3 + 8 \text{ is determined to be } 11$$

STEP 3



2 tens 1 ten and 1 one

STEP 4



3 tens 1 one
or
31

Either of the procedures described above could be used to assist a student who had one of the indicated error patterns for addition.

There are also many error patterns in the algorithm for subtraction:

Using Manipulative Aids to Model Algorithms in Remedial Situations

taking the "little number" from the "bigger number" even when regrouping is required, 23 :

$$\begin{array}{r} - 8 \\ 23 \\ \hline \end{array}$$

regrouping when it is unnecessary, 23 ; adding tens to the ones without

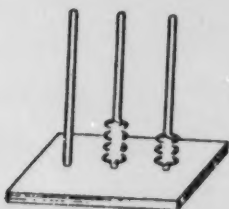
$$\begin{array}{r} - 2 \\ 23 \\ \hline 21 \end{array}$$

regrouping, 23 ; etc.

$$\begin{array}{r} - 8 \\ 23 \\ \hline \end{array}$$

As can be seen from the sample of error patterns above, place value knowledge or a lack of it seems to be important factor in performing the subtraction algorithm. Sets of counting sticks and centimeter rods can be used to illustrate the subtraction algorithm concretely so that a student might better understand the role of place value in subtraction, but the open-end abacus is helpful in this regard as well.

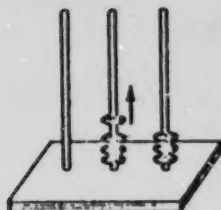
STEP 1



[Note: To show subtraction only
the sum need be shown.]

$$\begin{array}{r} 43 \text{ or } 4 \text{ tens } 3 \text{ ones} \\ - 8 \\ \hline \end{array}$$

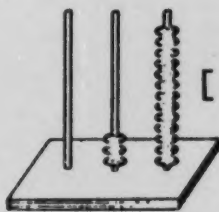
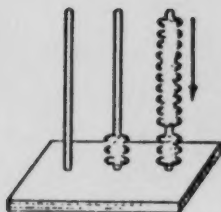
STEP 2



[8 cannot be subtracted from 3 in the set of whole numbers. so, rename the 43 or the 4 tens 3 ones as 3 tens + 1 ten and 3 ones.]

$$\begin{array}{r} 43 \\ - 8 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{r} 4 \text{ tens } 3 \text{ ones} \\ - 8 \text{ ones} \\ \hline \end{array} = \begin{array}{r} (3 \text{ tens} + 1 \text{ ten}) 3 \text{ ones} \\ - 8 \text{ ones} \\ \hline \end{array}$$

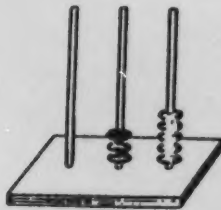
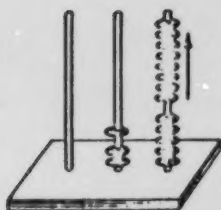
STEP 3



[Regrouping one ten as ten ones]

$$\begin{array}{r} 43 \\ - 8 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{r} 3 \text{ tens } 13 \text{ ones} \\ - 8 \text{ ones} \\ \hline \end{array}$$

STEP 4



$$\begin{array}{r} 43 \\ - 8 \\ \hline 35 \end{array} \quad \text{or} \quad \begin{array}{r} 3 \text{ tens } 13 \text{ ones} \\ - 8 \text{ ones} \\ \hline 3 \text{ tens } 5 \text{ ones} = 35 \end{array}$$

Using Manipulative Aids to Model Algorithms in Remedial Situations

With the open-end abacus students must be aware that a bead on the tens post is ten times as great in value as a bead on the ones post. If this is a difficult concept for a student to understand then the use of centimeter rods or counting sticks might be a better choice for modeling the subtraction algorithm because the concrete model for ten is visually larger than the one using these aids.

The complexity of the multiplication algorithm when w , two-place numbers are involved is often overwhelming to students. The process of computing four partial products and recording them in the proper place is very difficult. In fact some students who have reasonable understanding of place value often fail to do well with multiplication because the complexity of the algorithm causes them to abandon their knowledge for a rote method.

Centimeter rods offer us a unique method of illustrating the multiplication algorithm, and show each of the four partial products in an example such as, $12 \times 13 = \underline{\hspace{2cm}}$.

It may be helpful to remind students before using the rods that multiplication facts can be modeled with arrays.

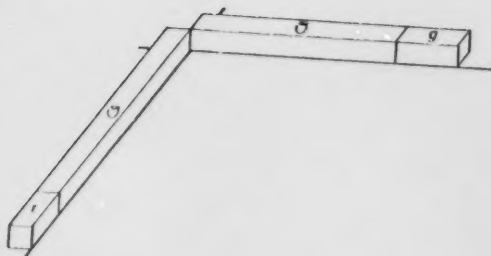
$$3 \times 4 = \underline{\hspace{2cm}}$$

•	•	•	•
•	•	•	•
•	•	•	•

3 rows of 4

With this knowledge, 12×13 can be set up as 12 rows of 13.

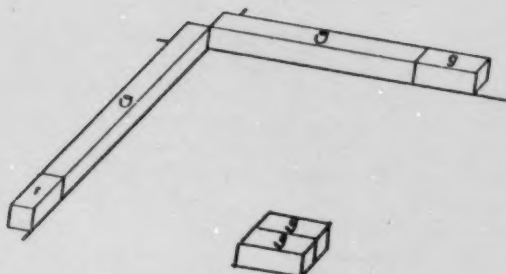
STEP 1



$$\begin{array}{r} 13 \\ \times 12 \\ \hline \end{array}$$

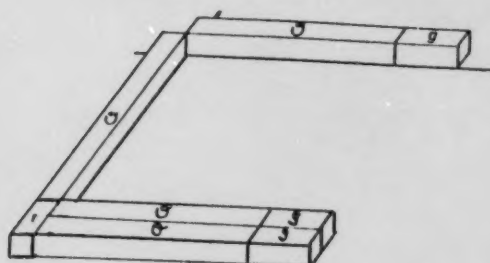
Hynes

STEP 2



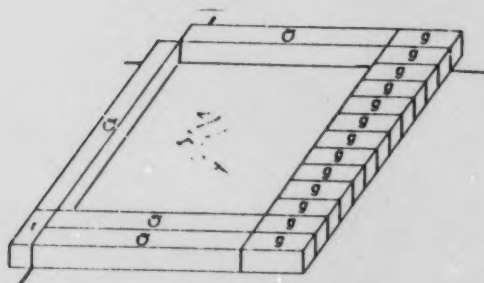
$$\begin{array}{r} 13 \\ \times 12 \\ \hline 6 \end{array}$$

STEP 3



$$\begin{array}{r} 13 \\ \times 12 \\ \hline 6 \\ 20 \end{array}$$

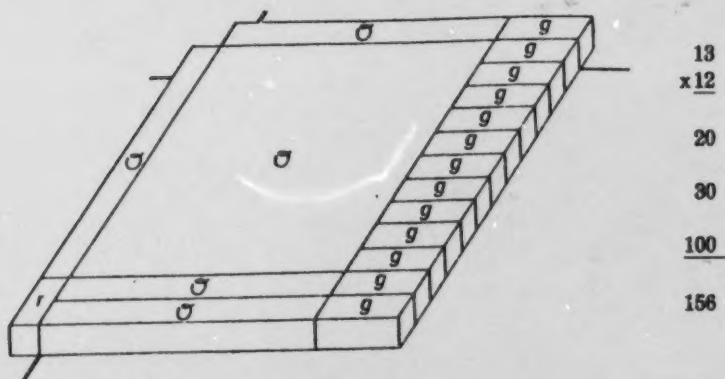
STEP 4



$$\begin{array}{r} 13 \\ \times 12 \\ \hline 6 \\ 20 \\ 30 \end{array}$$

Using Manipulative Aids to Model Algorithms in Remedial Situations

STEP 5



If students use this concrete model, they will be able to visualize why there are four partial products needed in the abstract algorithm. A better understanding of the multiplication process will result.

The division algorithm is even more complex than the multiplication one. Division requires the ability to add, subtract, multiply, compare numbers, and estimate--all in the proper sequence. It is no wonder that students make errors when faced with this complexity. Centimeter grid paper is one aid which may be used to illustrate the pyramid method of long division. To divide using the pyramid method, the following procedure is suggested.

4 $\overline{)152}$ is read, "How many sets of 4 are in a set of 152?"

Is there one set of 4 in a set of 152? - Certainly - $1 \times 4 < 152$

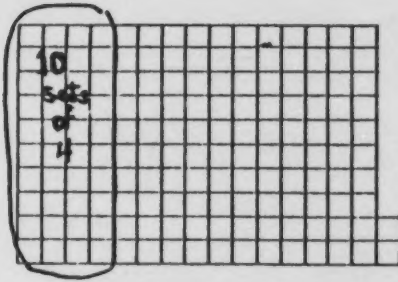
Are there ten sets of 4 in a set of 152? - Yes - $10 \times 4 < 152$

Are there 100 sets of 4 in a set of 152? - No - $100 \times 4 > 152$

Thus we know that the answer to this division example is in between ten and one hundred.

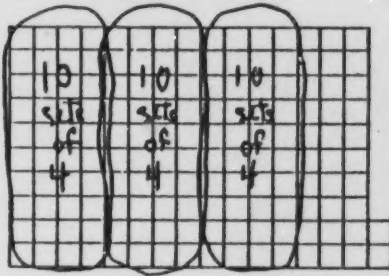
Let's remove ten sets of four squares from the set of one hundred fifty-two squares on the grid paper.

Hynes



$$\begin{array}{r} 10 \\ 4 \overline{) 152} \\ \underline{40} \\ 112 \end{array}$$

Can ten more sets of four be removed from the remaining squares? Yes, in fact, there are twenty more sets of four.

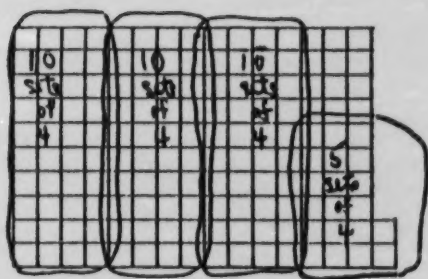


$$\begin{array}{r} 20 \\ 10 \\ 4 \overline{) 152} \\ \underline{40} \\ 112 \\ \underline{80} \\ 32 \end{array}$$

Can ten more sets of four be removed from the remaining squares? No, because there are only 32 squares remaining.

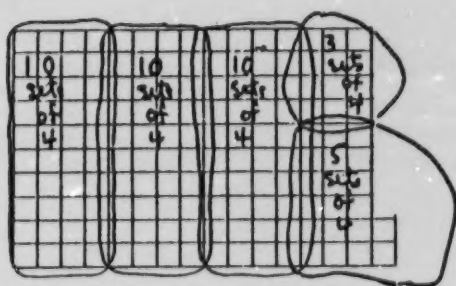
How many sets of four are in a set of 32? I am not sure, but I do know that $5 \times 4 = 20$. Let's remove 5 sets of 4.

Using Manipulative Aids to Model Algorithms in Remedial Situations



$$\begin{array}{r}
 5 \\
 20 \\
 10 \\
 4 \overline{) 152} \\
 \underline{40} \\
 112 \\
 \underline{80} \\
 32 \\
 \underline{20} \\
 12
 \end{array}$$

There are 12 remaining squares, and $3 \times 4 = 12$; so, the final 3 sets of four can be removed.



$$\begin{array}{r}
 3 \\
 5 \\
 20 \\
 10 \\
 4 \overline{) 152} \\
 \underline{40} \\
 112 \\
 \underline{80} \\
 32 \\
 \underline{20} \\
 12 \\
 \underline{12} \\
 0
 \end{array}$$

Hynes

The solution to the example is determined by merely adding up the partial quotients at the top of the algorithm. This model of division and the pyramid algorithm can help students understand the mystery of long division and reduce the number of clerical errors in this complex algorithm. Many examples of this type will help a student develop confidence in their ability to do mathematics and make them better mathematicians.

Throughout this paper, concrete models using manipulative aids were illustrated as solutions to overcoming a child's difficulties in mathematics. While it is true that manipulative aids and concrete models should be used in remedial situations, there are two points which must be made about the use of manipulative aids. First, the development of new concepts and algorithms should be taught on the concrete level to *all* children. Some children will proceed to discard the manipulative aids and work only on the abstract level very quickly. Others will cling to the "apron strings" of the manipulative aids, but all will benefit from having visualized the process. Secondly, even the remedial student should be encouraged to operate on the abstract level. Of course, this encouragement to throw away the manipulative aids must be tempered by reason. The whole point of using the manipulative aids is to help students gain a better understanding of and facility with mathematics. Thus, if a student really needs the aids let him feel that using the aids can be done without shame.

References

- Ashlock, R. M. Error patterns in computation: A semi-programmed approach, Columbus, Ohio: Charles Merrill. 1975.
- Hollingsworth, C. D. "The case of the ambiguous diagnosis", The FCTM Newsletter, 1977, 19 (2), 10.
- Hynes, M. C. "Response modes and diagnostic procedures: Needed research," In the Proceedings of the Third National Conference on Remedial Mathematics, (Edited by Heddens, J.W. and Acquila, F.D.) Kent State University, 1976.

A Theoretical Study of the
Organization of Basic Addition
Facts for Memorization

James W. Heddens

Research in the diagnosing of mathematics difficulties needs to be approached from many different perspectives. As Romberg and Uprichard (1977) suggested, "If one's knowledge about diagnosis and instruction in mathematics is to improve, then we collectively must be more imaginative in our approach to inquiry." Therefore, I would like to suggest one technique of approach to research and then discuss with you the first step in this particular process.

Review of the many papers presented at the first three national conferences on remedial mathematics indicates the importance that is placed upon the sequencing of mathematics into hierarchies. Because of the emphasis being placed on the importance of sequencing mathematics content and the encouragement to use creative research techniques, a four step procedure is being suggested. Initially a theoretical analysis is conducted from a logical mathematical approach. Second, the theoretical analysis needs to be verified by correlating the logical analysis with the actual learning patterns of children. The third step will be the synthesis of a diagnostic technique and a corrective teaching procedure that is compatible with the theoretical and technical analysis. A statistical analysis will need to be done to evaluate the effectiveness of the synthesized procedure as a final step.

With this four-step procedure in mind, I would like to discuss with you the *initial theoretical analysis* of a very troubled area, the addition basic facts. It is hoped that this analysis will lead to insight into why children have difficulty with the addition basic facts; why children have difficulty memorizing the basic addition facts; why children arrive in fifth, sixth or even higher grades and still do not have the basic addition facts memorized.

The initial approach in analyzing the basic addition facts is to define the basic addition facts. Examining the words "basic," "addition," and "fact" would seem to indicate that we are interested in the fundamental addition combinations that are the base upon which all other addition examples are constructed. Since there are ten basic digits and beyond the basic ten digits all numbers are written using two digits to express one number concept, then all basic facts would have one-digit only. Thus, addition examples with addends which have more than one-digit are not basic facts. For example

+3

three depends upon two basic addition facts, and the concept of place value. The two basic facts in this specific example are two add three and one add zero. Thus, twelve add three does not fit the definition of a basic addition fact as defined. From this discussion, it can be concluded that a basic addition fact

Heddens

is a one-place whole number added with a one place whole number in order to obtain a one or two-place whole number which is called the sum. All of the basic addition facts can be located by using an addition table and systematically identifying the basic addition facts. Place all possible one-place addends on both the horizontal and vertical axis of a rectangular table. Writing sums in each of the cells of the table created by the intersection of the addends on the horizontal and vertical axis generates what we call the addition table. The basic addition facts as defined above and developed through the table approach becomes as indicated below.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

It then becomes apparent that there are one hundred and only one hundred basic addition facts.

Using the addition table thus defined, there becomes two basic techniques which can be used to organize all the basic addition facts generated. One method, which seems to be most popular in elementary school textbooks, is to organize the basic facts according to the sums. What are all of the basic addition facts which have a sum of ten? Some textbooks call all of the combinations for a given sum a family of numbers. This defines nine basic addition facts that have a sum of ten and would be designated as the family of ten.

$$\begin{array}{cccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 +9 & +8 & +7 & +6 & +5 & +4 & +3 & +2 & +1
 \end{array}$$

Basic Addition Facts for Memorization

There is one example with a sum of zero, 0; two examples with a sum of one

$$\begin{array}{r} +0 \\ 0 \end{array}$$

1 and 0 ; three examples with the sum of two

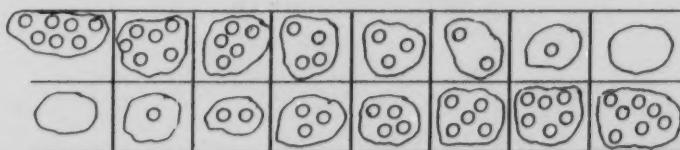
$$\begin{array}{r} +0 \\ 1 \end{array} \quad \begin{array}{r} +1 \\ 1 \end{array} \quad \begin{array}{r} +1 \\ 2 \end{array} \quad \begin{array}{r} +0 \\ 2 \end{array} \quad \begin{array}{r} +2 \\ 2 \end{array}$$

examine all of the families of numbers. The table summarizes this approach..

Sums	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Number of Examples	1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1

From the table above it becomes apparent that we have exhausted all possible examples; we have named all basic addition facts. Following the pattern of the table every cell to the left or right of the extended table would contain a zero for the number of examples. There cannot be any additional basic addition facts. Thus, the two tables above both suggest that there are only one hundred basic addition facts.

It becomes rather easy for young children to verify the basic facts as organized above by using sets of blocks, straws, or plastic spoons as counters. For example, if seven blocks are randomly tossed on a table in front of a child, he can easily separate the seven blocks into two separate sets and locate all of the possible combinations of two sets for the number seven.



This approach seems very logical for helping young children model the two sets of addends for a given sum. However, it does not seem to help the children with the concept of "addition" and it does not help the children memorize the basic addition facts. It does help the children develop a model to justify why a certain combination of two addends provides a unique sum. Most children who are taught by this approach depend upon counting for finding sums and become confused as to the difference between addition and counting. This method of organizing the basic addition facts becomes beneficial for development of addition through models but is of little value in helping memorize the basic addition facts. The common element in each of the

1
 2
 3
 4
 5
 6
 7
 8
 9
 10
 11
 12
 13
 14
 15
 16
 17
 18
 19
 20
 21
 22
 23
 24
 25
 26
 27
 28
 29
 30
 31
 32
 33
 34
 35
 36
 37
 38
 39
 40
 41
 42
 43
 44
 45
 46
 47
 48
 49
 50
 51
 52
 53
 54
 55
 56
 57
 58
 59
 60
 61
 62
 63
 64
 65
 66
 67
 68
 69
 70
 71
 72
 73
 74
 75
 76
 77
 78
 79
 80
 81
 82
 83
 84
 85
 86
 87
 88
 89
 90
 91
 92
 93
 94
 95
 96
 97
 98
 99
 100
 101
 102
 103
 104
 105
 106
 107
 108
 109
 110
 111
 112
 113
 114
 115
 116
 117
 118
 119
 120
 121
 122
 123
 124
 125
 126
 127
 128
 129
 130
 131
 132
 133
 134
 135
 136
 137
 138
 139
 140
 141
 142
 143
 144
 145
 146
 147
 148
 149
 150
 151
 152
 153
 154
 155
 156
 157
 158
 159
 160
 161
 162
 163
 164
 165
 166
 167
 168
 169
 170
 171
 172
 173
 174
 175
 176
 177
 178
 179
 180
 181
 182
 183
 184
 185
 186
 187
 188
 189
 190
 191
 192
 193
 194
 195
 196
 197
 198
 199
 200
 201
 202
 203
 204
 205
 206
 207
 208
 209
 210
 211
 212
 213
 214
 215
 216
 217
 218
 219
 220
 221
 222
 223
 224
 225
 226
 227
 228
 229
 230
 231
 232
 233
 234
 235
 236
 237
 238
 239
 240
 241
 242
 243
 244
 245
 246
 247
 248
 249
 250
 251
 252
 253
 254
 255
 256
 257
 258
 259
 260
 261
 262
 263
 264
 265
 266
 267
 268
 269
 270
 271
 272
 273
 274
 275
 276
 277
 278
 279
 280
 281
 282
 283
 284
 285
 286
 287
 288
 289
 290
 291
 292
 293
 294
 295
 296
 297
 298
 299
 300
 301
 302
 303
 304
 305
 306
 307
 308
 309
 310
 311
 312
 313
 314
 315
 316
 317
 318
 319
 320
 321
 322
 323
 324
 325
 326
 327
 328
 329
 330
 331
 332
 333
 334
 335
 336
 337
 338
 339
 340
 341
 342
 343
 344
 345
 346
 347
 348
 349
 350
 351
 352
 353
 354
 355
 356
 357
 358
 359
 360
 361
 362
 363
 364
 365
 366
 367
 368
 369
 370
 371
 372
 373
 374
 375
 376
 377
 378
 379
 380
 381
 382
 383
 384
 385
 386
 387
 388
 389
 390
 391
 392
 393
 394
 395
 396
 397
 398
 399
 400
 401
 402
 403
 404
 405
 406
 407
 408
 409
 410
 411
 412
 413
 414
 415
 416
 417
 418
 419
 420
 421
 422
 423
 424
 425
 426
 427
 428
 429
 430
 431
 432
 433
 434
 435
 436
 437
 438
 439
 440
 441
 442
 443
 444
 445
 446
 447
 448
 449
 450
 451
 452
 453
 454
 455
 456
 457
 458
 459
 460
 461
 462
 463
 464
 465
 466
 467
 468
 469
 470
 471
 472
 473
 474
 475
 476
 477
 478
 479
 480
 481
 482
 483
 484
 485
 486
 487
 488
 489
 490
 491
 492
 493
 494
 495
 496
 497
 498
 499
 500
 501
 502
 503
 504
 505
 506
 507
 508
 509
 510
 511
 512
 513
 514
 515
 516
 517
 518
 519
 520
 521
 522
 523
 524
 525

THE





11

1
2
3
4
5
6
7

Basic Addition Facts for Memorization

Note that there are nineteen basic addition facts that can be related to the identity element of addition. Thus remembering one rule, the identity element of addition, children can recall nineteen basic addition facts or about one-fifth of them.

Another set of basic addition facts may be related to the addition of one or counting. Consider the examples below.

$$\begin{array}{cccccccccccccccccccc}
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\
 +1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +9 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
 \hline
 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3
 \end{array}$$

How are all of these examples alike? How are all of these examples different? One of the addends in each example is one. The sum is one greater than the other addend. Children can generalize that adding one with a number gives the next number. This is just like counting. These examples can be located in the second row and the second column of the addition table.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

← Second Row

↑ Second Column

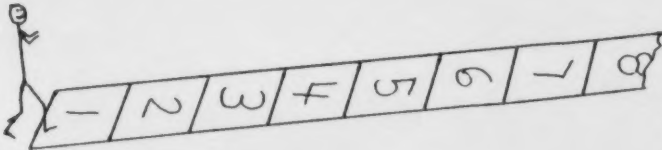
Note that there are seventeen basic addition facts that can be related to the idea of counting. If children can relate counting to these facts then they will have these seventeen basic addition facts memorized.

There are nineteen basic addition facts that can be identified with the identity element of addition and seventeen facts can be related to counting.

Heddens

Thus, thirty-six basic addition facts, or more than one-third of the basic addition facts have been memorized by understanding and remembering two rules.

Let us now examine another subset of the basic addition facts. The children should have had readiness experiences during the past year leading to this subset of addition facts. Children should have had the experiences of walking on a number line and saying the numbers upon which either the right or the left foot steps. For instance the child might hit his right leg with his right hand as he walks on the number line and says that number as he steps on it. The same type of activity should be done using the left leg and the left hand.



After this type of real world experience, the children should be asked to count by twos. Also one must remember to tell the child with which number to begin counting. When counting by twos 1, 3, 5, 7, 9, 11, 13,... is just as valid as 2, 4, 6, 8, 10, 12,.... The child should be able to begin at 4 or 7 any other number within the child's range of understanding. This subset of basic addition facts would be related to the concept of twoness. Examine this set of fifteen basic addition facts.

2	3	4	5	6	7	8	9	2	2	2	2	2	2	2
$\frac{+2}{4}$	$\frac{+2}{5}$	$\frac{+2}{6}$	$\frac{+2}{7}$	$\frac{+2}{8}$	$\frac{+2}{9}$	$\frac{+2}{10}$	$\frac{+2}{11}$	$\frac{+9}{11}$	$\frac{+8}{10}$	$\frac{+7}{9}$	$\frac{+6}{8}$	$\frac{+5}{7}$	$\frac{+4}{6}$	$\frac{+3}{5}$

Note that this set of basic addition examples is located in the addition table in the third column and the third row.

Basic Addition Facts for Memorization

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Third Row

Third Column

Children should be encouraged to discover what ideas they observe from this set of examples. They can generalize that it is just like counting by two or that when two is added with another number you just skip a number. Three skip four and say five, thus three add two is five.

Placing these fifteen addition examples with the thirty-six addition facts already developed, we now have a total of fifty-one basic addition facts developed and memorized. The children only need to remember three basic rules or ideas and yet they have fifty-one basic addition facts memorized. Is it easier to remember fifty-one isolated facts or is it easier to remember three rules and relate the fifty-one examples to the rules? As children grow, develop, discover, and generalize, each class of examples becomes more difficult to generalize a rule.

The next set of addition facts that we will examine let's classify as the doubles. There are seven of these examples and they are located in the addition table on the diagonal. These examples are:

3	4	5	6	7	8	9
$\frac{+3}{6}$	$\frac{+4}{8}$	$\frac{+5}{10}$	$\frac{+6}{12}$	$\frac{+7}{14}$	$\frac{+8}{16}$	$\frac{+9}{18}$

Heddens

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Children seem to have less difficulty memorizing this set of addition facts than many of the other addition facts. Since these seem to be easily memorized, we use this as a base for the set of basic addition facts which we call the related double. These examples are listed below:

Doubles	3	4	5	6	7	8	9
	$\begin{array}{r} +3 \\ 3 \\ \hline 6 \end{array}$	$\begin{array}{r} +4 \\ 4 \\ \hline 8 \end{array}$	$\begin{array}{r} +5 \\ 5 \\ \hline 10 \end{array}$	$\begin{array}{r} +6 \\ 6 \\ \hline 12 \end{array}$	$\begin{array}{r} +7 \\ 7 \\ \hline 14 \end{array}$	$\begin{array}{r} +8 \\ 8 \\ \hline 16 \end{array}$	$\begin{array}{r} +9 \\ 9 \\ \hline 18 \end{array}$
	$\swarrow \searrow$	$\swarrow \searrow$	$\swarrow \searrow$	$\swarrow \searrow$	$\swarrow \searrow$	$\swarrow \searrow$	$\downarrow$
Related	3 4	4 5	5 6	6 7	7 8	8 9	No
Doubles	$\begin{array}{r} +4 \\ 7 \end{array}$ $\begin{array}{r} +3 \\ 7 \end{array}$	$\begin{array}{r} +5 \\ 9 \end{array}$ $\begin{array}{r} +4 \\ 9 \end{array}$	$\begin{array}{r} +6 \\ 11 \end{array}$ $\begin{array}{r} +5 \\ 11 \end{array}$	$\begin{array}{r} +7 \\ 13 \end{array}$ $\begin{array}{r} +6 \\ 13 \end{array}$	$\begin{array}{r} +8 \\ 15 \end{array}$ $\begin{array}{r} +7 \\ 15 \end{array}$	$\begin{array}{r} +9 \\ 17 \end{array}$ $\begin{array}{r} +8 \\ 17 \end{array}$	Related Doubles

Note that there are twelve related doubles and they are located in the table on two diagonals.

Basic Addition Facts for Memorization

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

The sum for each related doubles is one more than the sum for the double.
Since 3 then 3 and 4.

$$\begin{array}{r} +3 \\ 6 \end{array} \quad \begin{array}{r} +4 \\ 7 \end{array} \quad \begin{array}{r} +3 \\ 7 \end{array}$$

Yes, this is more abstract but the children are becoming older and more acquainted with developing and using generalizations.

Children may use the doubles, associative property of addition and adding one to a number as a technique of recognizing the sum. For example:

$$\begin{array}{lcl} 3 + 4 & = \square & \\ 3 + (3 + 1) & = \square & \text{Rename} \\ (3 + 3) + 1 & = \square & \text{Associative property of addition} \\ 6 + 1 & = \square & \text{Adding doubles} \\ 7 & = \square & \text{Adding one} \end{array}$$

This provides real power in mathematics for each child. The seven doubles and twelve related doubles, along with the fifty-one facts already developed makes a total of seventy basic addition facts.

Heddens

Children should be developing an understanding of tenness and place value. The children can bring together the concept of addition and the concept of place value for the set of basic addition facts which we will call the basic facts related to tenness. There are ten of these facts.

3	4	5	6	7	9	9	9	9	9
$\frac{+9}{12}$	$\frac{+9}{13}$	$\frac{+9}{14}$	$\frac{+9}{15}$	$\frac{+9}{16}$	$\frac{+7}{16}$	$\frac{+6}{15}$	$\frac{+6}{14}$	$\frac{+4}{13}$	$\frac{+3}{12}$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

After studying these examples children should generalize it will take one with the set of nine to make a set of ten. The set of ten is placed in the tens place, leaving one less than the number being added in the ones place. Thus we see that in the example $9 + 5 = \square$ one from the five is placed with the nine to make ten. The example becomes $10 + 4$ or fourteen. With a good place value understanding children have little difficulty discovering and applying this generalization of tenness.

Place these ten examples with the seventy already developed we have a total of eighty basic addition facts. That means there are twenty basic addition facts yet to be learned. The commutative property of addition will reduce to ten the number of basic addition facts to learn. These are the difficult examples for children to learn because there is no unifying pattern. Each child is encouraged to search for ideas that will provide individually help. the last set of facts is:

Basic Addition Facts for Memorization

5	6	7	8	6	7	8	7	8	8
$\frac{+3}{8}$	$\frac{+3}{9}$	$\frac{+3}{10}$	$\frac{+3}{11}$	$\frac{+4}{10}$	$\frac{+4}{11}$	$\frac{+4}{12}$	$\frac{+5}{12}$	$\frac{+5}{13}$	$\frac{+6}{14}$

and the commutative property provides the other ten

3	3	3	3	4	4	4	5	5	6
$\frac{+5}{8}$	$\frac{+6}{9}$	$\frac{+7}{10}$	$\frac{+8}{11}$	$\frac{+6}{10}$	$\frac{+7}{11}$	$\frac{+8}{12}$	$\frac{+7}{12}$	$\frac{+8}{13}$	$\frac{+8}{14}$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Sometimes the comment is heard that if the children understand the commutative property of addition that this will cut the number of basic facts in half. This is not a true sentence because the facts we have classified as doubles cannot be commuted. Forty-five of the basic addition facts can be commuted and ten of the basic facts cannot be commuted. The child still must memorize fifty-five of the basic addition facts.

In working with young children and the basic addition facts it seems advisable to develop the basic addition facts in families of numbers and memorize the facts according to structures. (A family of numbers is all of the pairs of addends that have a given sum.) If a child is memorizing the basic addition facts by using structures he is not learning the facts as specified. Instead of memorizing one hundred unrelated facts he is memorizing structures and then placing addition examples into the structures.

Heddens

References

- Grossnickle, F. E. and Brueckner, L. J. Discovering meanings in elementary school mathematics. Philadelphia: John C. Winston Company, 1959.
- Heddens, J. W. Today's Mathematics, Third edition. Palo Alto, California: Science Research Associates, 1974.
- Romberg, T. A. and Uprichard, A. E. The nature of the clinical investigation, A paper presented at the Fourth National Conference of Remedial Mathematics, University of Maryland, 1977.
- Thiele, C. L. The contributions of generalizations to the learning of the addition facts. New York: Bureau of Publications, Teachers College, Columbia University, 1938.

How Do You Get Kids To Quit Using Their Fingers To Add And Subtract?

Edward C. Rathmell

Given the current interest in Chisanbop, perhaps the title needs an explanation. This paper is not addressed to that method of computation, but rather to the children who consistently and persistently count their fingers one by one to solve addition and subtraction facts which should be memorized. Teachers find these tedious counting mechanisms often obstruct the normal progress that a child should make in computation.

Examine the paperwork of the five children below:

Mary	2	4	9
	$\frac{+4}{6}$	$\frac{+5}{9}$	$\frac{+3}{12}$

Bill	2	4	9
	$\frac{+4}{6}$	$\frac{+5}{9}$	$\frac{+3}{12}$

Jody	2	4	9
	$\frac{+4}{6}$	$\frac{+5}{9}$	$\frac{+3}{12}$

Chris	2	4	9
	$\frac{+4}{6}$	$\frac{+5}{9}$	$\frac{+3}{12}$

Pat	2	4	9
	$\frac{+4}{6}$	$\frac{+5}{9}$	$\frac{+3}{12}$

Would you teach these children the basic addition facts using the same instructional strategies? Perhaps it would help to know the amount of time that it took each child to complete the problems.

Rathmell

Mary 2 4 9 20 seconds

$$\begin{array}{r} +4 \\ 6 \end{array} \quad \begin{array}{r} +5 \\ 9 \end{array} \quad \begin{array}{r} +3 \\ 12 \end{array}$$

Bill 2 4 9 14 seconds

$$\begin{array}{r} +4 \\ 6 \end{array} \quad \begin{array}{r} +5 \\ 9 \end{array} \quad \begin{array}{r} +3 \\ 12 \end{array}$$

Jody 2 4 9 33 seconds

$$\begin{array}{r} +4 \\ 6 \end{array} \quad \begin{array}{r} +5 \\ 9 \end{array} \quad \begin{array}{r} +3 \\ 12 \end{array}$$

Chris 2 4 9 20 seconds

$$\begin{array}{r} +4 \\ 6 \end{array} \quad \begin{array}{r} +5 \\ 9 \end{array} \quad \begin{array}{r} +3 \\ 12 \end{array}$$

Pat 2 4 9 22 seconds

$$\begin{array}{r} +4 \\ 6 \end{array} \quad \begin{array}{r} +5 \\ 9 \end{array} \quad \begin{array}{r} +3 \\ 12 \end{array}$$

This may provide some additional clues as to the differences among the children, but do you have useful information for instruction? Examine the chart below to see how each child explains answers to these problems.

Pupil	$2 + 4 =$	$4 + 5 =$	$9 + 3 =$
Mary	Immediate response of 6. Explained that you count, four--five, six.	Said, "Nine, 4 and 4 is 8, so this is 1 more."	Thought for a while, then said, "Nine and 1 is 10, and 2 more is 12."
Bill	Said: "Six, because $4 + 2$ is 6."	"Well, 5 and 5 is 10, so 1 less is 9."	"That's hard. Let's see, 10 and 3 is 13 and 1 less is 12."
Jody	Said six, then explained by putting up two fingers on one hand and on the other hand counting three, four, five, six.	Put up five fingers on one hand and said, "I've got 4, so five, six, seven, eight, nine." as she counted her fingers.	"I'd need blocks for that." Then counted nine blocks and three blocks, combined them, and counted by ones to 12.

How Do You Get Kids To Quit Using Their Fingers To Add And Subtract?

Chris	"That's cinchy. It's 6." The explanation was, "I just know."	Put up four fingers on one hand and counted five, six, seven, eight, nine on the other hand.	Counted nine fingers by ones, then counted on to ten, thought a bit, and said, "eleven, twelve."
Pat	Said: "Two and two is four, and two more is six."	Thought a bit, then said: "Nine, because 3 and 5 is 8 and 1 more is 9."	Hesitated, then said: "Well, 3 and 3 is 6, and 6 more is 12."

CHART 1

After seeing the explanations that the children have given, do you feel that you have a better idea about the instructional needs of these pupils? Far more valuable information is supplied in this last chart.

We need to make an effort to determine how children are thinking about basic facts just as we do for computational algorithms. If a child is having difficulties with renaming in subtraction, our first thought is to find out what is wrong. We try to determine how the child is thinking. On the other hand, if a child is having difficulties with the basic facts, what do we do? The traditional response is to assume the student needs drill. Perhaps in recent years the approach would be to provide activities with concrete materials.

Sometimes these solutions are effective, sometimes they are not. Before you decide what instructional activities to use, it is helpful to know how that child is thinking. Perhaps what the child needs is drill. Perhaps the child needs work with concrete models. Perhaps the child needs to develop more efficient ways to think about the basic facts.

The expectations for learning basic facts should include two aspects. First, students should be expected to memorize basic facts so they have immediate recall. Second, they should be able to explain how to get the answers. Further examination of the chart on pupil thinking indicates some of the variety of different thinking strategies which can be used as explanations.

Most of us agree on the first expectation. The second one is presented for two reasons. First, research evidence indicates that helping children learn ways to figure out answers to basic facts other than counting on their fingers helps them learn and remember the basic facts. Second, the explanations help provide teachers continual feedback about the thinking that students are using. This helps teachers plan more appropriate instructional activities.

With these two expectations in mind, there seem to be three types of activities which are helpful in order for children to learn the basic facts.

1. During the initial learning about an operation, the use of concrete models helps students develop meaning for appropriate language and symbolism. They also help students relate the operation to the real world.
2. Next, it is helpful to the students to learn new and more efficient thinking strategies than counting on their fingers. When they count on their fingers they start right at the beginning for each new problem. It speeds them up considerably to learn to use what they already know each time, rather than start each problem from scratch. The thinking strategies also provide a relationship among facts that helps the child organize them into a meaningful whole so they do not have to memorize one hundred independent facts. This is a tremendous aid for retention.
3. The last type of activity that is needed is drill. It should be used to develop speed and accuracy. Drill does not change the way a child is thinking. It only speeds up the thinking they are using.

The remainder of this presentation is devoted to the second of these, helping students develop new thinking strategies. Of the three types of activities, it is the most likely to be left out of instructional programs and it is the most likely to be an aid to children who count on their fingers. How can you help children develop new patterns of thinking about basic facts? It doesn't happen by one lesson explaining it. Children need to practice new thinking strategies just as they need to practice a computational algorithm. This even becomes more difficult for students who have a long history of using an ineffective method of solving problems. Changing the way a child thinks involves breaking a habit.

One technique that has been effective and easy to implement is simply to spend a few minutes each day to review the thinking pattern you are trying to help a child learn. Perhaps this might be only three to five minutes each day. After about two weeks of these short activities most children are using the new thinking strategies spontaneously.

These short activities can mainly be oral combined with manipulation of concrete models. This is a very non-threatening way to keep the new thinking patterns visible to the child for a period of time that is long enough for him to internalize it.

The following activities are samples of the type of short three to five minute instructional activities. These have successfully been used with a variety of students. Teachers have used these with individuals and small groups of students with learning problems. They have also been used in self-contained classrooms with the entire class.

Addition Strategies

Counting On

Counting on has proven to be an effective and efficient way to add when

How Do You Get Kids To Quit Using Their Fingers To Add And Subtract?

one of the addends is small. If one addend is three or less, children are able to count on the keep track of how many they have counted without using their fingers.

Children who have a tendency to count from one each time can be taught to use counting on if they are confronted with a situation where counting from one is not possible. This can be done by hiding the greater set.

Example: Show dot cards with numerals on the back.



"How many?" (6)



"How many here?" (2)

Turn the "6" card around.



"How many in all?"

Prompt, if necessary.

"6,...,7,8."

Turn the card around again and count from 1 to check.

"1,2,3,4,5,6,7,8."



One More or One Less

Relating a fact problem to one that is known is also an effective way to help children answer basic facts. These known facts are quite often the doubles, but not always.

Example:



"What problem is shown?" (4+5)

"What double could we use to help us?" (4+4)

Remove one counter.

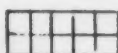


"How does that help?"
(4+4 is 8 so 4+5 is just 1 more or 9.)

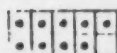
Add to Ten

When one of the addends is an 8 or 9, it is often effective to add to ten, then add the remainder. This requires that students can add a single-digit number to ten quickly and easily. Some practice is necessary to insure that students recognize the ease of this task.

Example:



"Show the problem $9+4$."



"How can we make this an easy problem?" (make 10)

Move a counter over to make a set of 10.



"9 and 1 is 10, and 3 more is 13."

Adding with Endings

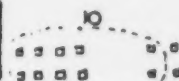
Even after learning addition and subtraction facts, some students resort to finger counting to solve multiplication facts. For example, " 3×6 is 18, so 4×6 is 18, ..., 19, 20, 21, 22, 23, 24." An addition strategy that has been found to help these students is adding with endings. This has been particularly effective where students start with a known fact to obtain an unknown fact in the next decade.

Adding with endings involves two ideas. First, recognizing what decade the sum is in, and second, knowing the ending (one's place) of the sum.

Example:



"What problem is shown?"
($28 + 4$)



"Is the sum in the twenties or thirties?" (thirties)
"How do you know?"
($8+4$ is more than 10)

Thirty - two

$$28 + 4 = 32$$

"What is the ending?" (2)
"How do you know?"
($8+4=12$ which ends in 2.)

How Do You Get Kids To Quit Using Their Fingers To Add And Subtract?

After this skill is learned, it can be used to solve any multiplication fact.

Example:

$$6 \times 7 =$$

"What is 6×7 ?" (not known)
"What fact might help?"
($5 \times 7 = 35$ so 6×7 is
 $35 + 7$ or 42.)

Subtraction Strategies

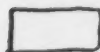
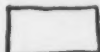
Relate to Addition

Children who are having trouble memorizing the subtraction facts often use a "count back" or "count what is left" strategy. These strategies work fairly well with small numbers, but not so well with facts like $13 - 8$. Encourage children to make use of the addition fact they already know. This can be done by using take away and hiding the set that is remaining.

Show 8 objects on the overhead projector.



Use an index card to cover them.



Write $8 - 3 =$ ____

Write $3 + 5 = 8$

Uncover the objects to check. Show the parts of 3 and 5 to reinforce the connection to addition.



"How many are there?" (8)

"I'm going to cover them. How many are covered?" (8)

"Let's take 3 out from under the card. What number sentence could we write to show that we had 8 under the card and removed 3?" ($8 - 3 =$ ____)

"There is an addition problem which can help us solve this problem. What number can be added to 3 to get 8? (5) A group of 3 and a group of 5 make 8. How many do you think are under the card? (5) What is $8 - 3$?" (5)

Three comments about the use of models are appropriate. All involve the thinking strategies that children learn when they work with models.

1. When children use models, they learn thinking strategies whether or not we specifically teach them. Chances are that students with learning problems will learn to use inefficient counting strategies if they are not taught more effective strategies. Persistent use of inefficient strategies contributes to their learning difficulties.
2. How models are used is just as important as whether they are used. If children are continually permitted to use inefficient counting strategies with models, they are not learning appropriate thinking for later symbolic work. In fact, they may be acquiring a counting habit that is difficult to change.

For example, most children use the number line for addition by counting "jump" from the first addend. They then check the number beneath the number line to determine the sum. This strategy works fine as long as the student uses a number line. However, the thinking that is being used is not helping the child learn to solve the problem when the number line is not present.

3. Using concrete to abstract learning sequences may be necessary for meaningful learning, but they are certainly not sufficient. Meaningful connections between model manipulation and symbol manipulation are made by developing thinking strategies which are efficient for both levels of representation. These strategies help children bridge the gap between concrete and abstract.

Summary

Many children count on their fingers to add and subtract. They will probably continue to use that inefficient method until they learn a better way to do it. The strategies presented here have been used with children to help them learn basic thinking skills so counting on their fingers is not necessary.

When activities to develop efficient thinking have been combined with drill, the results have been good. Students with learning problems have made gains in both learning and retention. If you have a student having difficulty learning the basic facts, find out how the child is thinking. You may be able to help a student avoid finger counting.

But I Have 42 Students
In My Class

Ralph D. Connelly

Diagnosis in mathematics has been receiving a great deal of attention recently. Diagnostic clinics are springing up all over the place, and those fortunate enough to have attended RCDPM conferences over the last four years have been treated with a barrage of information relative to pupil difficulties in mathematics. There is no doubt that the clinical setting, with a one-to-one testing situation conducted by skilled diagnosticians is our single best source of insights into how pupils learn (or mis-learn) mathematical concepts.

Yet with all the activity, with all the information we have being generated, many classroom teachers seem to feel that there is no way of applying this information to their classroom. "*But I have 42 students in my class*". The main purpose of this session is to try to give a few suggestions as to how this application might be accomplished. If you're a classroom teacher, I hope you'll find the suggestions useful. If you're a "diagnostician" who has heard this phrase many times, hopefully this will give you some ideas for a response.

First, evidence from the clinics seems to be pointing to one fact that really sets the whole *attitudinal* and *procedural* stage for what follows:

MOST PUPIL ERRORS IN COMPUTATION ARE SYSTEMATIC (i.e., the pupil applies an algorithm erroneously *consistently*.)

If teachers did nothing more than accept that statement, with its corresponding implications for diagnostic procedure, what a giant step it would be! In other words, as a first step, try to become familiar with some of the common error patterns that exist. It's impossible to know them all (particularly since pupils are coming up with new ones all the time), but there are many "popular" ones that we can learn to recognize. Ashlock's Error Patterns in Computation remains a monument in this area. Let's take a moment to look at a few examples.

An "easy one":

$$\begin{array}{r} 47 \\ + 28 \\ \hline 615 \end{array}$$

$$\begin{array}{r} 26 \\ + 57 \\ \hline 713 \end{array}$$

Very common (why)?:

$$\begin{array}{r} 34 \\ \times 7 \\ \hline 358 \end{array}$$

$$\begin{array}{r} 26 \\ \times 3 \\ \hline 78 \end{array}$$

Connelly

How about division?:

$$\begin{array}{r} 44 \\ 2 \overline{) 88} \\ \underline{80} \\ 8 \\ \underline{8} \\ 0 \end{array}$$

$$\begin{array}{r} 14 \\ 4 \overline{) 164} \\ \underline{160} \\ 4 \\ \underline{4} \\ 0 \end{array}$$

$$\begin{array}{r} 67 \\ 3 \overline{) 228} \\ \underline{21} \\ 18 \\ \underline{18} \\ 0 \end{array}$$

$$\begin{array}{r} 39 \\ 5 \overline{) 465} \\ \underline{45} \\ 15 \\ \underline{15} \\ 0 \end{array}$$

(right to left,
like other operations)

Are you ready for a toughie?:

$$\begin{array}{r} 233 \\ 2 \overline{) 176} \\ \underline{4} \\ 16 \\ \underline{16} \\ 0 \end{array}$$

$$\begin{array}{r} 221 \\ 4 \overline{) 824} \\ \underline{8} \\ 24 \\ \underline{24} \\ 0 \end{array}$$

$$\begin{array}{r} 231 \\ 3 \overline{) 713} \\ \underline{6} \\ 13 \\ \underline{12} \\ 1 \end{array}$$

$$\begin{array}{r} 213 \\ 3 \overline{) 639} \\ \underline{6} \\ 39 \\ \underline{39} \\ 0 \end{array}$$

(greater no. + lesser,
ignore remainder)

OK experts, a quick look at fractions- Oh, I know you all know $1/4 + 2/3 = 3/7$ but how about:

$$\frac{1}{4} + \frac{2}{3} = \frac{6}{12} + \frac{8}{12} = \frac{14}{12} \quad \frac{2}{3} + \frac{1}{2} = \frac{4}{6} + \frac{3}{6} = \frac{7}{6}$$

and $12\frac{1}{8} = 11\frac{9}{8}$

$$3\frac{1}{3} = 3\frac{2}{6} = 2\frac{12}{6}$$

$$-5\frac{7}{8} = -5\frac{7}{8}$$

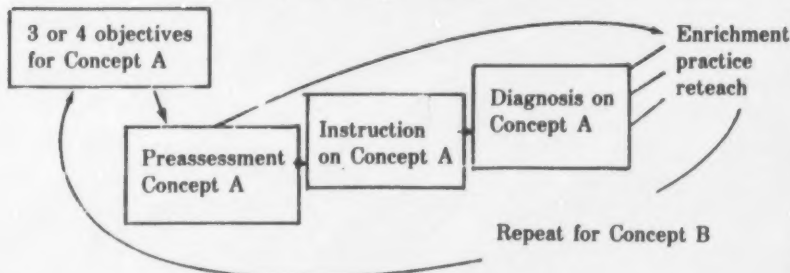
$$-1\frac{1}{2} = -1\frac{3}{6} = -1\frac{3}{6}$$

$$\underline{\hspace{1cm}} \quad 6\frac{4}{8} = 6\frac{1}{2}$$

$$\underline{\hspace{1cm}} \quad 1\frac{9}{6} = 2\frac{3}{6} = 2\frac{1}{2}$$

Enough for now. Step one completed- so now we're experts at recognizing error patterns- So what? *I still have 42 students in my class!*

Forgive me now for going theoretical, just for a moment. A couple of years back, Robert Underhill (1976) suggested a model for classroom diagnosis in a paper given at a RCDPM meeting. I've tried it and it does work (not ideal mind you, but it works.) Very briefly, the model looks like this:



42 Students in My Class

To apply the model is not difficult, but it does make certain assumptions. First that objectives for the concept are available; some method of preassessment is available some method of diagnosis is available. Let's deal with these separately to show that the task is not overwhelming.

Objectives: School District Guidelines (example for Washington District Schools in Phoenix)
Textbook Scope and Sequence charts (available free from publishers)
KSU (Kent State University) Math team's checklist (available from KSU)
Other checklists (example of my own- Many available-conference people are great sharers)

Preassessment: You will likely make your own, and I would recommend this, but some overall diagnostic tests are available which can be effectively used with the "whole class" to preassess year's work in advance. Dr. Lelon Capps (University of Kansas) prepared a list of available diagnostic tests in mathematics (and analyzed many of them) for the 1977 conference.

Diagnosis: Clearly same test *could* be used, but I wouldn't suggest this. What I *would* suggest is a different use of homework exercises and homework marking time. Particularly at the junior high level, and often at the 4-6 level as well, homework is marked by just putting a check in the corner of the paper and occasionally X'ing out wrong answers. What I would suggest is a simplified "diagnostic" approach to homework assignment and checking:

- 1) Select exercises carefully to reflect specified objectives. The odd-numbered exercises on the bottom of page 62" do not guarantee the meeting of this condition. Yes, teacher-made sheets would be more effective here, but even a more cautious selection of exercises from p. 62 might suit your needs.

- 2) It is vital to stress an orderly arrangement of work, and that all work be shown.

- 3) Look for error *patterns*, not just errors. Even if this means a detailed look at only a *fourth* of the papers and a cursory glance at the others, over the course of a week's homework you'll have done *some* diagnosis of each pupil's work. This is far from ideal, but it's a start. Besides, we all have some pupils who we know will not be having any difficulties (perhaps fewer than we'd like, but we have *some*) and we can therefore devote more of our limited time to those whom we expect might be having greater difficulties.

- 4) Write it down *somewhere*. Don't trust your memory to remind you that "I must work with Jimmy on his division this week." If a checklist for

Connelly

each child (which is what I would strongly suggest) seems unreasonable or unwieldy to you, then how about a folder for each child where, when a problem is diagnosed, you write "RETURN" on the "offending" homework and after the pupil has been made aware of his errors, the paper goes into his file folder (and can be removed, if desired, when the problem is remediated.)

This "homework diagnosis" clearly deals only with the abstract level of Underhill's model, but again, it's a start.

We can also get diagnostic information from the weekly (or whatever interval) quizzes we construct. Because they tend to be more cumulative than our assignments, we can use them to get an idea of where in the sequence of a concept the pupil is having difficulty and the nature of the difficulty. This information can be recorded at the same time the test score is being recorded. To effectively use quizzes for this purpose, the same steps as for homework need to be followed. Note that the *time* for constructing and grading the quizzes is not significantly greater, it's just that the *focus* of these procedures is altered from what is common practice.

If you're willing to use "off-the-cuff" diagnostic information, a surprisingly fertile source is the math activity corner of your room. This may seem like a very unlikely place to organize diagnostic procedures, but some of my best insights into pupil difficulties have come from this source. Nowhere is a pupil freer with his "patented algorithmic shortcuts" than in the heat of a math game competition. To use your activity corner for this purpose, any of your game boards will do. Number the problem cards and again concentrate on *sequence* and *objectives* when designing the cards. When pupils are using the game, have them number the problems as they do them, then just have them leave their sheets there when they're finished. "In your copious free time" look through the papers—you'll likely find some interesting results (Note: a course in hieroglyphics helps this process.) The main reason I like this source, "slipshod" as it is, is that it may have certain attitudinal benefits. That is, when a pupil feels he is being "tested," diagnostically or otherwise, there is sometimes a tendency for the pupil to withdraw; to simply not do the problems, sometimes leading to the diagnosis that he doesn't have any idea about the concept at all. The "game" setting removes that atmosphere and can lead to a clearer indication of the pupil's difficulties.

I have not considered remediation in this presentation at all. However, I am not so naive as to believe that just getting more diagnostic information from techniques like those mentioned today will alleviate pupil difficulties to any great degree. Clearly remediation in the classroom setting presents an equal, if not greater, challenge than does diagnosis and hopefully ideas in this regard will be forthcoming. I *do* believe, though, that we often get insights into pupils' thinking patterns from the diagnostic information we gather, and that in many cases these insights lead to adjustments in our teaching strategies so that others are not led into the same erroneous thought patterns. That's a start!

Implementing Diagnosis In The Classroom

James E. Inskip, Jr.

The current emphasis of diagnosis and prescription for mathematics difficulties has done little to alleviate the classroom teacher's work load. The learning clinic seems to be one solution in dealing with the problems of individual children. However, few teachers can afford the luxury of time needed to implement a clinical approach. Making and implementing plans for individualization of instruction, decreasing the pupil-teacher ratio by including para-professionals and cross age (or peer) "teachers", and just extra expenditure of time and work have been partial responses to the need for individual diagnosis and prescription.

Another approach to the problem is to analyze current procedures and schedules to develop more efficient methods of diagnosis. Many classroom teachers use paper and pencil type tests as a means for evaluating progress and difficulties children have with mathematics. In this paper we will focus attention upon: (1) the development of effective diagnostic tests which may be used in place of other paper and pencil tests, and (2) the analysis of these tests to aid in developing individual and/or small group approaches to particular problems in computation. The focus is on computation because it is an important segment of mathematics instruction and is frequently evaluated with paper and pencil tests.

Consider the following flow chart which diagrams an approach to the teaching of mathematics in the classroom. Note that this chart includes pre- and post-test components which some teachers may not use. Even with the absence of such tests, most teachers use some sort of evaluation, quite often employing the common paper and pencil test.

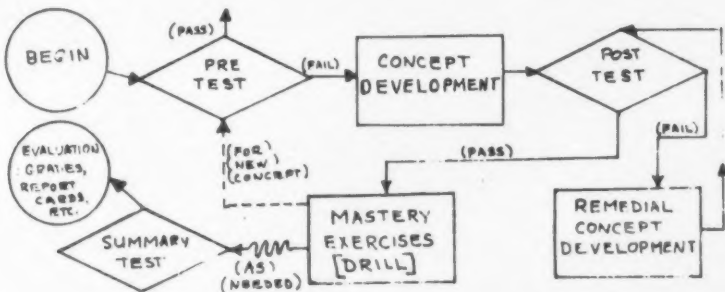


Chart 1

It is to the paper and pencil evaluation (pre-test, post-test, or just "test") that we will direct our comments. In the previous flow chart there are three points at which a diagnostic test can be used. Diagnosis may be made at all three points or at any one.

A good diagnostic test should include most of the following qualities to make it effective for correct and efficient diagnosis:

1. The test pinpoints specific errors and difficulties.
2. Evaluation for prescription and remediation is based upon an adequate display of the problem(s) given to the child (i.e. there are enough instances of error to give weight to a decision made.).
3. There is an economy of time and management incorporated into the construction, evaluation and use of the test.
4. Opportunity for interpretation and evaluation of the test results is simply and adequately provided.
5. No information which would normally be available to the teacher on a paper and pencil test is lost by making a special diagnostic instrument.

The suggestions for construction, interpretation, and records for a diagnostic test meet the above noted criteria.

Development of the Diagnostic Test

The first step in developing a good diagnostic test is to decide the content of the test. In a clinical situation, the clinician will normally give some sort of survey test and then narrow the diagnosis to specific areas which appear to need critical evaluation. The classroom teacher, on the other hand, will be working with a given segment of content and already has the scope narrowed to that which is to be learned by a group or individual child. For children who have been exposed to instruction but have not learned the content, the diagnosis will be undertaken in a specific area.

The actual construction of the test should follow the following steps:

1. Make a list of the computational behaviors you wish to test, using using either the conventional behavioral objective form or a series of examples illustrating the behavior.
2. With the computational behavior in mind, use three examples which do not differ from each other in any significant way.
3. Mix up your examples, coding them appropriately to indicate the type each represents and make out your test.

An illustration of these steps follows.

Test Construction Illustrated

Step 1: Behaviors (Only three will be used here; a longer test may be appropriate for most needs.)

Type 1: Multiplication: one-digit number times two-digit number, with regrouping.

Diagnosis In The Classroom

Type II: Multiplication: one-digit number times three-digit number, with regrouping from one's to ten's place and ten's to hundred's place.

Type III: Multiplication: one-digit number times three-digit number (with zero in ten's place), with regrouping from one's to ten's place.

Step 2: Examples Step 1 can be skipped if the examples are chosen to reflect particular behaviors.)

Type I:	$\begin{array}{r} 17 \\ \times 6 \\ \hline 102 \end{array}$	$\begin{array}{r} 33 \\ \times 8 \\ \hline 264 \end{array}$	$\begin{array}{r} 53 \\ \times 9 \\ \hline 477 \end{array}$	Code: See below in step 3. ()
Type II:	$\begin{array}{r} 134 \\ \times 7 \\ \hline 938 \end{array}$	$\begin{array}{r} 175 \\ \times 6 \\ \hline 1050 \end{array}$	$\begin{array}{r} 246 \\ \times 8 \\ \hline 1968 \end{array}$	/ /
Type III:	$\begin{array}{r} 309 \\ \times 5 \\ \hline 1545 \end{array}$	$\begin{array}{r} 206 \\ \times 6 \\ \hline 1236 \end{array}$	$\begin{array}{r} 406 \\ \times 3 \\ \hline 1218 \end{array}$	--

Step 3: Construction (Examples above taken in mixed order and coded for identification by type.)

(1)	/2/	(3)
$\begin{array}{r} 17 \\ \times 6 \\ \hline \end{array}$	$\begin{array}{r} 309 \\ \times 5 \\ \hline \end{array}$	$\begin{array}{r} 33 \\ \times 8 \\ \hline \end{array}$
-4-	/5/	/6/
$\begin{array}{r} 206 \\ \times 6 \\ \hline \end{array}$	$\begin{array}{r} 134 \\ \times 7 \\ \hline \end{array}$	$\begin{array}{r} 175 \\ \times 6 \\ \hline \end{array}$
(7)	-8-	/9/
$\begin{array}{r} 53 \\ \times 9 \\ \hline \end{array}$	$\begin{array}{r} 406 \\ \times 3 \\ \hline \end{array}$	$\begin{array}{r} 246 \\ \times 8 \\ \hline \end{array}$

The administration of the diagnostic test should incorporate these two aspects:

1. As the children (group, class, or individual) take the test observe them, noting specific work habits or counting procedures which may be visible.
2. Do not impose strict time limits but allow each child to finish, as is practicable. In other words, do not test for overlearning at this point.

Basic fact difficulties are in a class by themselves. If you do not have some other method of testing for basic facts, make up such a test as a companion for your diagnostic test. The basic facts test should be timed. The primary objective in giving this type of test is to determine which of the basic facts give difficulty. Consequently, it makes sense to include only the basic facts which will be needed for your particular diagnostic test. The basic facts test may be given at any point in the instructional sequence but the information should be available to you when you go to analyze your diagnostic test. Also, it is wise not to include basic facts on your diagnostic test. The basic facts should probably be tested against time constraints while the diagnostic test should not. Note that basic fact difficulty will tend to hide or mask error patterns, leading you to think that the errors are random. In some cases, the only difficulty a child has is with the basic facts. The problem however is that many teachers tend to blame *all* computational errors on lack of mastery of basic facts and this is not necessarily true.

Interpretation of Results

Having given your test to your children, there remains the task of analysis and interpretation. Since you would normally mark paper and pencil tests, recording the results, there is only one additional task needed to make your test diagnostic. You need to make a specific analysis of the errors made. Some of the specific kinds of things we would expect to observe on our diagnostic test are:

1. *Distinct error patterns* indicating that the child has made an error of procedure or concept which is a consistent and learned behavior for that child.
2. *Random errors*. This type of error persists to adulthood even for those with good computational skills. (At this point note the rationale for using three examples of a given type. The use of three examples enables the teacher to have: (1) a sample of more than one on which to base diagnosis, (2) enough data to note error patterns, and (3) a basis for defining what a random error is (i.e. one error out of three).)
3. *Omitted examples or no response*. This type of behavior is often considered to be just an error. Actually, there is neither error nor correct response. Some children know what they do not know and do not care to guess. Other children may know but are timid and do not feel secure in responding. Remediation for this problem will take a different form from that for an error pattern.
4. *Guessing errors* or those which have no pattern. Some children feel the need to answer each and every example on a test whether they know it or not. These responses will follow no distinguishable pattern but it is well to note that some patterns are very sophisticated and elude even the sharpest pattern-conscious teacher. When in doubt, ask

Diagnosis in the Classroom

the child to show you how he did the examples. This procedure will not need to be done with a large number of children. Also, be alert to the fact that specific basic fact difficulties will "muddy the waters" and lead you to think that the errors have no pattern. When in doubt check the records as to the particular basic facts which might affect the child's responses.

5. Evidence of *counters*. Often, a child who is a counter will use some paper and pencil method for making his counts. Of course, you may also observe this behavior as he takes the test and it would be wise to not this. Children count in many creative ways!

Records and Management

Little has been said about the management of a diagnostic paper and pencil test program since most teachers will desire to use their own system. The following comments are designed to give you some suggestions for improving or possibly establishing a good system of management and records. The suggestions deal with the marking of the tests as well as general ideas for records.

1. Make your analysis of each test on the paper. Mark up the copies directly. You may wish to use a system of shorthand or code which cues you as to the type of error and/or error pattern made. As you make your analysis, do more than just mark incorrect examples with an "X" or circle. Mark each of the examples for a given type together and make notes. Above all, be alert to error patterns. A prescription or suggestion for remediation may be made on the back of the test paper or included on the front with a different colored marker.
2. Don't hand back the tests you intend to use for diagnosis. Keep them and maintain a file for each of your children or at least for those children who have consistent difficulty. Note that you do not have to make diagnostic tests in place of every pre-test, post-test, and other test that you give. However, you may wish to cut down on the amount of testing you do by regularly substituting diagnostic tests in place of summary evaluative tests. You can make up grades on the basis of your diagnostic test results *and* you will have good information for report cards, parent conferences, and individual conferences with the child.
3. Record keeping can be done by means of files or by check sheets which list all of your computational behaviors. The types of records used with many of the individualized programs are nicely suited to this approach. The records you keep are your own and should be made to work for you. In some cases, merely keeping the diagnostic tests will be a good record of your children's progress and difficulty.
4. Remediation will follow at the teacher's discretion and should be keyed to the difficulties found. The scope of this presentation does not allow

Inskeep

for many suggestions for remediation. The following brief suggestions are just hints.

- (a) Alert your para-professional(s) to the diagnostic tests giving them instructions as to what remedial measures to take (which can be noted on the papers).
- (b) Investigate alternative methods of presenting particular content. If kits are available which can be used on an individual basis, use them as needed. Games and other game-like materials often can be used when the diagnosis points toward the need for drill or re-introduction.
- (c) Handle drill work with care. Practice can be very detrimental to computational skill, particularly when applied to children who evidence error patterns. Use with care and only when you know that the basic concepts and procedural steps for the algorithms are understood. With every drill activity, provide immediate or near-immediate feedback. Give answers for drill; withhold them for diagnostic testing!

George A. Wyer

A Fundamental and Costly Flaw

Of growing concern in the teaching of mathematics to kindergarten through grade eight children today is the increasing inability to provide opportunities for *complete* concept acquisition. Symptomatic of this problem is excessive use of paper and pencil "workbook" exercises and testing too often carried out only at the symbolic level of abstraction. The problem is compounded by delays in initiating concept development at the optimum grade level as suggested by the syllabus.

The many pressures contributing to this situation include: the "knowledge explosion", pressure to "mainstream" handicapped children, and pressure to accommodate a myriad of special programs and services. The result is a teacher and child trapped beneath an expanding curriculum while slowly being buried by snow balling content deficiencies which continue to accumulate for future, presumably, "individualized" remediation.

Previous Solutions - A Mistaken Emphasis

Previous efficiencies used to treat this problem include: individualizing instruction, forming homogeneous groups or classes, and integrating subject matter. In the case of total reliance on individualized instruction, gains would seem to be more than offset by losses from not treating grade-level content using whole class instruction. In regard to homogeneous grouping, a review of the literature by Findley and Bryan (1970) found considerable evidence that ability grouping, usually based on standardized tests, was generally ineffective in improving academic achievement and, in fact, tended to "maintain and increase class stratification in our society." Ability grouping is also reported to have detrimental effects particularly on low-ability groups (Heathers, 1969). Forming homogeneous (ability) classes is supposed to minimize the programming demands on the teacher. This may well be the case when developing subject matter in *adults*. However, *homogeneous ability groups/classes are of little practical value when treating the diversified needs and varying cognitive growth rates of the developing child*. What works for adults is not necessarily good for children.

Consider, for example, that the development of mathematics for adults is virtually the reverse of the sequence in which math concepts are acquired by the developing child (Aiken, 1977; Wyer, 1972). Also, *specialization* of content is the norm in developing subject matter for adults. But to be consistent with the way in which concepts are acquired by the developing child, it would follow that *integration* of content would be more suitable. However, as with individualized instruction, used exclusively, integration of content could result, at

best, only in a different set of problems with no real gain.

A relatively recent set of efficiencies for consideration in improving learner performance relate student learning style and cognitive style, teacher style, and teaching method within the instructional program. R. Dunn and K. Dunn (1979), in describing 18 elements of learning style, present a strong argument against trying to match student with teachers claiming instead that "most teachers can respond to differences in student learning styles." In identifying various dimensions of cognitive style, Mullally (1977, p. 238) lists the following:

1. Field independent/dependent (Witkin)
2. Focusers/scanners (Bruner)
3. Descriptive/relational/categorical elements (Kagan)
4. Convergent/divergent/evaluative types of cognitive operations (Guilford)

Each of these is a dimension of the developing child upon which one might be tempted to base an instructional program.

While it will be important to consider all of the aforementioned efficiencies in planning an instructional program, I strongly caution against program planning that is polarized around any one particular efficiency such as: individualized vs. whole class instruction, competitive vs. individualistic goal structures, or field dependent vs. field independent cognitive style. Such dichotomies should be avoided (NACOME, 1975, p. 136) for they cannot by themselves provide a basis for program to accommodate the total needs of the developing child. We recognize that a child's growth involves the social, affective, and psychomotor, as well as the cognitive domain. The question remains: How can programming for mathematics instruction account for the total needs of the child across all these domains?

This dilemma is best resolved using a programming vehicle which accommodates both the treatment of grade-level content as well as the treatment of the child's individual enrichment and remedial needs and does so without compromising the integrity of the concept development process. Such a vehicle I call Dual Programming.

What Is Dual Programming?

Dual Programming is an instructional plan that incorporates individualized treatment of children's enrichment as well as remedial needs along with whole class treatment of grade level content. This is accomplished in a setting which I believe maximizes opportunity for a child's overall social, affective, psychomotor, and cognitive development. For this the teacher must demonstrate skill in: (a) using intra-class flexible grouping strategies, (b) using diagnostic/prescriptive techniques, and (c) providing for proper concept development.

Dual Programming



Individualized Program

In the individualized segment of Dual Programming children from a given group, out of perhaps three permanent "social" groups in the class, are interviewed one or two at a time in a low-key, personalized and supportive setting. During the interview a child can demonstrate mastery of mathematical skills and concepts directly to the teacher while also providing a range of cues to aid in further prescription. The two remaining groups are occupied in non-teacher-directed activities relating primarily to the grade level program.

The systematic and relatively independent occupation of these remaining groups is necessary for the success of Dual Programming. This is accomplished by selecting interest-heightening activities involving conceptually powerful materials. Such materials should allow for a wide range of problem solving strategies and abilities so that each child's experience can be intellectually and socially productive.

Grade Level Program

In the grade level segment of Dual Programming, content typically developed in the grade the child is enrolled in is introduced using a variety of small group "hands on" activities under minimal teacher supervision. Several weeks later the concepts/skills imbedded in these experiences may be extracted through whole class discussion and at a higher level of abstraction.

Wyer

The development is extended and reinforced in the usual way through application and take-home prescriptions. Thus, the development of content appropriate for a particular grade level is begun at that grade level regardless of whether mastery at the symbolic level of abstraction is imminent.

Evaluation of student learning of grade level content may be accomplished using a combination of pencil-pushing (show-all-your-work) and multiple choice tests. Concepts/skills not mastered are targeted for prescription under the individualized program.

Flexible Grouping

It generally takes several weeks to ready a class for cycling groups. The orientation should stress:

1. involvement in cooperation activities.
2. learning to care for materials.
4. understanding the need for shared responsibility in a group.
5. cooperative student/teacher formation of groups.

I call number 5 (above) "social contract grouping" and recommend this form of homogeneous intra-class grouping over ability grouping or heterogeneous (often defined as "random") grouping. Also, my experience indicates the optimum *weighted* class size (Note 1) for Dual Programming K-8 is 18 to 21 students.

Now, assume that a class of 26 students has been partitioned into three groups. A typical weekly schedule for cycling would then look like this:

Figure 2
Sample Grouping Schedule
TUESDAY WEDNESDAY THURSDAY

Group Activity i.e. "chip trading" (two setups)	A	C	B
<u>Prescription Group</u> TEACHER DIRECTED (...individual work on skill tapes, calculator problem solving, etc.)	B	A	C
Group Activity ie. "angle measurement" (with partner)	C	B	A

A,B,C = groups of 8 or 9 students each

Note. Monday and Friday - whole class instruction

Dual Programming

Assuming 40 to 50 minute class periods, the first ten or fifteen minutes of class before "groups" may be used for whole class instruction since most group activities can be planned for under thirty minutes. The options for cycling groups in this fashion are many and varied. Try it!

Diagnosis and Prescription

The initial diagnosis. Within the first few days of meeting any new group of students it is wise to administer a short pencil-pushing screening test that samples prerequisite mathematical concepts/skills. This will: (a) suggest adjustments to the anticipated "grade level program", (b) provide handwriting samples, (c) indicate sophistication of problem solving strategies and algorithm development, and (d) contribute to a more productive and personalized first interview. Do this even if your school administers standardized norm-referenced tests later on. Aside from the additional waiting time, such tests do not, in general, provide handwriting or problem solving strategy information.

"Prescription group" interview. Flexible grouping provides the teacher with the opportunity to interview and observe students on a one-to-one basis. Received *directly* and immediately, the resulting data reflects the whole child in relationship to his mathematical growth, as opposed to the *indirect*, delayed, and impersonal feedback characteristic of paper-and-pencil, consider-the-answer-only tests administered en masse.

For the latter a *wrong* answer without the work could be the result of a random error requiring only a short review or it could also result from a lack of understanding of the concept/skill and thus require careful systematic development. A *correct* answer without the work gives no indication of the level of maturity of the problem solving strategy or algorithm employed. In either case we would be hard-pressed to make a prescription based on such unreliable information. Increasing the number of test items for a particular concept might help in the case of a "wrong" answer but not in the case of a "correct" answer.

If such indirect testing predominates, it is because of a lack of opportunity to deal directly with the student in the first place.

"Prescriptive group" time does permit the teacher to diagnose and prescribe directly for each child based on:

1. information obtained from the previous interview (as recorded in the student's class notebook which should not leave the room).
2. student concerns and questions relating to homework, quizzes, class lectures, etc.
3. concepts/skills not mastered under the grade level program.
4. schoolwide "diagnostic" testing.
5. observation of that child particularly while engaged in small group problem solving activities.
6. feedback from class discussion.

And because this time concludes with a bridge to further mathematical growth

Wyer

in the form of a take-home prescription or, perhaps, another diagnosed need, the prescription group interview process is open-ended.

Evaluation is accomplished by demonstrating mastery on one or two problems during an interview. Mathematical growth can be recognized by assigning units of credit on a *progress chart*. No one has "failed" an interview yet!

Proper Concept Development

Dual Programming permits: (a) involvement at all levels of the concrete to abstract continuum and in correct temporal sequence, (b) visual, aural, and kinesthetic experience to the degree required by the child, (c) sensitivity to a wide range of learning styles, and (d) a balance of "cooperative" as well as "competitive" and "individualistic goal structures" (D. Johnson and R. Johnson, 1975). In short, it permits *complete* development of grade level as well as individualized enrichment and remediation needs.

Summary

Dual Programming is inspired by and responds to the needs of the developing child. It is *not* a product of the development of mathematics, nor is it based on some particular facet of mathematics instruction or student learning style. Its singular importance to mathematics learning and instruction is that, using direct observation and interview as opposed to indirect paper-and-pencil tests, individual diagnosis and prescription is made possible. And, the same flexible grouping strategy that permits this clinical capability in the classroom also supports complete concept development. Using Dual Programming is similar to finding a cure for the common cold. No longer must you wait for the child to develop pneumonia before you can effectively treat him.

Footnote

¹In determining class size a learning disabled child might be counted ("weighted") as two children, a severe remedial child as one and a half, and so on. The "weighting" of class size takes on considerable importance in, for example, the "mainstreaming" of exceptional children.

Dual Programming

References

- Aiken, L.R., Jr. Mathematics as a creative language. Arithmetic Teacher, 1977, 3, 251-255.
- Dunn, R. S. and Dunn, K. J. Learning styles/teaching styles: Should they...can they...be matched? Educational Leadership, 1979, 4, 238-244.
- Findley, W. G. and Pryan, M. M. Ability grouping: 1970 -- II. The impact of ability grouping on school achievement, affective development, ethnic separation and socioeconomic separation. Georgia University, Athens, Georgia, College of Education, 1970.
- Heathers, G. Grouping. Encyclopedia of Educational Research, Fourth Edition, New York: The MacMillan Co., 1969, 559-570.
- Johnson, D.W. and Johnson, R. T. Learning together and alone. New Jersey: Prentice-Hall, Inc., 1975.
- Mullally, L. Educational cognitive style: Implications for instruction. Theory Into Practice, 1977, 16, 238-242.
- National Advisory Committee on Mathematical Education. Overview and analysis of school mathematics grades K - 12. Washington, D.C.: Conference Board of the Mathematical Sciences, 1975.
- Wyer, G. A. Hypothesized inhibiting factors in the mathematics training of prospective elementary school teachers. Paper presented at the 2nd International Congress on Mathematics Education, Working Group No. 26, Exeter, England, September, 1972.